Symmetry Axiom and Haar Measure for Random Order Shapley Value of Games *

Dr. Lakshmi K. Raut
Visiting Fellow,University of Chicago
Center for the Economics of Human Development
1126 E. 59th St., Chicago, IL 60637,USA
mailto:lakshmiraut@gmail.com

Abstract

Imposing a group structure on the set of random orders, the paper reformulates and characterizes the random order value and more generally semi-value operators in a unified framework that encompasses games with finite and continuum of players and allows symmetry of the operators to be with respect to a subgroup of automorphisms. A set of orderings of players equipped with a group structure induced from the group structure of automorphisms together with a measure structure on it constitutes a group of random orders in the analysis. For finite games it is shown that given any fixed group of random orders, the linear operator on the whole space of games that assigns to each game its expected marginal contribution is symmetric with respect to the associated group of automorphisms if and only if the randomness of the group of orders is generated by a right invariant Haar measure; as a corollary, the paper provides a group theoretic proof for the existence and uniqueness of random order Shapley value and semi-value operators that are symmetric with respect to the full group of automorphisms; the paper also shows that the random order semi-value operators constructed from a proper subgroup of orders coincide with the semi-value operators which are symmetric with respect to the full group of automorphisms on a linear subpace of games. Many of these results are also extended to games with continuum of players.

JEL Classification Number: C71
Keywords: Shapley Value, Random Order, Non-Atomic Games, and Haar Measure.

*This paper is based on my Indian Statistical Institute Discussion paper #8105, June 1981, New Delhi. I am grateful to Llyod Shapley who introduced me to game theory, and to K.R. Parthasarathy who introduced me to measure theory. Comments from Donald Brown and Maxwell Stinchcombe were very useful in revising the draft.

1 Introduction

Given a fixed player set (known as grand coalition) and an algebra of subsets of the player set (denoting the set of possible coalitions), a cooperative game is a set function that assigns the worth to each coalition which the players in the coalition can earn cooperatively. One of the basic problem in cooperative game theory is to find rules for dividing the worth of the grand coalition among its members such that the solution has certain nice properties. More precisely, the problem is to find a mapping of a set of cooperative games to the set of finitely additive set functions such that the mapping possesses certain properties. Shapley value is one among many such solution concepts as core and stable set.

Shapley value proposed by Shapley, 1953 for finite games and extended by Aumann and Shapley, 1974 to the games with continuum of players, and the semi-value proposed by Dubey et al., 1981 for games with finite as well as continuum of players have many interesting economic applications. For instance, in the case of increasing returns to scale in production, marginal cost pricing and average cost pricing are respectively characterized in terms of Shapley value and semi-value (see for instance, Samet and Tauman, 1982 for a survey, and Billera et al., 1978 which was the first attempt to use Shapley value for measure valued games with a continuum of players to telephone billing problem). In recent years, Shapley value and semi-value have been reinterpreted in many ways, see contributions in Roth, 1988 for an account of this.

The set of cooperative games with a fixed finite player set forms a finite dimensional vector space. Shapley, 1953 postulated that the mapping from the space of all games to the space of finitely additive set functions be linear, symmetric with respect to the group of automorphisms of the player set, and efficient (definitions are in subsequent sections). This mapping is known as axiomatic Shapley value operator. He showed that a certain class of simple games forms a basis of the linear space of all games and for each simple game in the basis, there exists a unique finitely additive set function that satisfies the axioms. Utilizing the linear vector space structure of the space of games he proved the existence and uniqueness of the axiomatic Shapley value operator on the whole space of games and provided a formula to compute the value of individual games. Shapley also provided an alternative approach to the above basic problem of cooperative game theory which later came to be known as random order approach. In this approach, a player is given his expected marginal contribution in a random order of players, each order being equally likely among all possible orderings of the players. He showed that the formula for expected marginal contribution of players coincide with the axiomatic value formula for every game.

Aumann and Shapley, 1974 extended the concept of axiomatic value for games with continuum of players and by utilizing certain topological properties such as internality of the underlying linear vector spaces of games, they proved the existence and uniqueness of axiomatic value operator on the spaces, bv′NA and pNA (definitions are in subsequent sections). A semi-value is a value without efficiency axiom. A parallel analysis of axiomatic semi-value for both finite and continuum case has been carried out by Dubey et al., 1981. In an attempt to extend the random order approach to values of games with continuum of players, Aumann and Shapley, 1974 proved the impossibility of finding a probability measure structure on the set of orders of the grand coalition that could yield a random order value operator on reasonable spaces of games. They proposed instead the concept of mixing value as a close substitute of their random order value and investigated the existence and characterization of mixing value.

In certain applications, the value and more generally semi-value operators might be required to be symmetric with respect to only a proper subgroup of the group of all automorphisms. For instance, we may consider the group of those orders in which a set of players stay together. In Raut, 1981 and in the present paper, we reformulate and extend the random order value approach of Shapley, 1953 and Aumann and Shapley, 1974 in an unified framework encompassing games with finite and continuum of players as follows:

In finite players games, each automorphism of the player set generates a distinct order of the players. A set of orders with a measure structure is referred as the set of random orders. We show that given any fixed group of random orders, the linear operator on the whole space of games that assigns to each game its expected marginal contribution is symmetric with respect to the group of associated automorphisms if and only if the randomness of the group of orders is generated by a right invariant Haar measure; moreover, a semi-value is a value if right invariant Haar measure is a probability measure. Utilizing the Haar measure theorem (Halmos, 1950, Theorem B, p.254, Parthasarathy, 1977, Proposition 54.2), we prove that given a fixed group of orders, there exists one dimensional linear space of (resp. unique) random order semi-value operators (resp. value operator). The random order semi-value operators are convenient way to generate linear operators on the whole space of games that are symmetric with respect to a fixed subgroup of automorphisms, although the space of such operators need not be one dimensional unless the symmetry is with respect to the full group of automorphisms. As a corollary, we provide a proof for the existence and uniqueness of random order Shapley value and semi-value operators symmetric with respect to the full group of automorphisms, which utilizes only the group structure of the set of automorphisms and parallels the proof of the uniqueness and existence result for axiomatic Shapley value and axiomatic semi-value operators.

We furthermore show that given a proper subgroup of random orders, there exists a linear subspace of games on which the random order semi-value operator with respect to the proper subgroup of random orders coincides with a random order semi-value operator with respect to the full group of random orders. Thus a random order semi-value (resp. value) operator with respect to a proper subgroup is also a semi-value (resp. value) operator with respect to the full group of symmetries for certain linear subspaces of games. In general, we cannot, however, expect that the linear subspaces of games on which these two operators coincide are symmetric with respect to the full group of automorphisms.

We also extend the above random order approach to continuum case. Most of the results for finite case extend to continuum case provided we restrict the fixed subgroup of orders to complete and second countable metric groups that are locally compact in the case of random order semi-value operators and compact in the case of random order value operator. Since in the continuum case, subgroups of orders that are derived from automorphisms could be uncountably large, the existence of Haar measure on them is not trivial. Using techniques from the theory of group representations, we also provide a general procedure to construct topological group of random orders. In this paper, we provide examples of large but finite group of orders with respect to which the random order value exists for a large class of games. In Raut, 1997, we extend Daniel-Kolmogorov consistency theorem of stochastic processes to construct an uncountably large group of random orders admitting Haar probability measure and we have also shown that the corresponding random order value operator yields the same diagonal value formula of Aumann and Shapley for a certain class of scalar and vector measure games in pNA.

Section 2 deals with the random order value and semi-value operators for games with finite set of players. Section 3 deals with the continuum case.

2 Games with finite number of players

2.1 Background

Let $N = \{1, 2, \cdots, n\}$ be a fixed set of n players and let $\mathcal{P}_N$ be the power set of N. We will denote the real line by $\Re$. Let $G_N = \{V : \mathcal{P}_N \to \Re \mid V(\emptyset) = 0\}$ be the set of all *games*. One can verify easily that G_N is a linear vector space over $\Re$ of dimension $2^n - 1$.

Let $FA = \{V \in G_N \mid V$ is a finitely additive set function $\}$. Let $\Theta = \{\theta : N \to N \mid \theta$ is one-one and onto$\}$ be the set of automorphisms of the player set N. It is easy to note that Θ is a group with composition of maps as the group operation. For each $\theta \in \Theta$, define the linear operator $\theta^* : G_N \to G_N$ by

$$(\theta^* V)(S) = V(\theta^{-1} S)$$

Note that the above definition of θ^* is slightly different from what is used in Aumann and Shapley [1974, p.15]. However, since θ is 1-1 and onto there is no real difference between two definitions. Let Q be a linear subspace of G_N. We denote by $Q^+ = \{V \in Q \,|S, T \in \mathcal{P}_N, S \supset T \Rightarrow V(S) \geq V(T)\}$. The games in Q^+ are said to be *monotonic games* in Q. Q is *symmetric* with respective to group Θ if for all $\theta \in \Theta$, $\theta^* Q \subset Q$. Let $\Phi : G_N \to FA$ be an operator. For any $\theta \in \Theta$, we define the operator $\theta^*\Phi : G_N \to G_N$ by $(\theta^*\Phi)(V) \equiv \theta^*(\Phi(V))$.

An operator $\Phi : Q \to FA$ is said to be *linear* if $\Phi(\alpha V_1 + V_2) = \alpha\Phi(V_1) + \Phi(V_2)\ \forall\, V_1, V_2 \in Q, \alpha \in \Re$. On a symmetric space Q, Φ is said to be *symmetric* if $\Phi\theta^* V = \theta^*\Phi V,\ \forall\ \theta \in \Theta, V \in Q$. Φ is said to be *positive,* if $\Phi Q^+ \subset FA^+$. Φ is said to be *efficient* if $\Phi V(N) = V(N)\ \forall\ V \in Q$. A player $i \in N$ is a *null player in game V* if $V(S \cup \{i\}) = V(S) + V(\{i\})$ for all $S \in \mathcal{P}_N$ such that $i \notin S$. The symmetry of $\Phi : Q \to FA$ with respect to Θ has the following commutative diagram:

$$\begin{array}{ccc} Q \ni V & \overset{\Phi}{\longmapsto} & \Phi V \in FA \\ \theta^* \downarrow & & \downarrow \theta^* \\ Q \ni \theta^* V & \underset{\Phi}{\longmapsto} & \theta^*\Phi V \in FA \end{array}$$

A linear, symmetric operator on a symmetric subspace Q is said to be a *semi-value operator* on Q. An efficient semi-value operator on Q is said to be a *value operator*.

Consider the following *simple* games,

$$V_R(S) = \begin{cases} 1 & if\ S \supseteq R \\ 0 & otherwise \end{cases} \tag{1}$$

Shapley, 1953 [1] showed that the set of games, $\{V_R | R \in \mathcal{P}_N, R \neq \emptyset\}$ forms a basis for G_N, and each game in the set has a unique value characterized by the three axioms of Shapley

[1] For alternative expositions of the above argument, see Aumann and Shapley, 1974 or Shapley and Raut, 1981.

value operator; using the linear vector space properties of G_N, and the linearity of Φ, proved that there exists a unique Shapley value operator Φ on G_N and is given by

$$(\Phi V)(\{i\}) = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!}[V(S) - V(S - \{i\})], \tag{2}$$

where, $s = |S|$. Shapley has shown that a value operator is always positive, and that if $i \in N$ is a null player in game $V \in Q$, then $(\Phi V)(\{i\}) = V(\{i\})$.

Shapley, 1953 gave a bargaining model later known as *random order approach*. We follow Aumann and Shapley, 1974 to describe it.

An *order* on N is a transitive, irreflexive and complete binary relation $\succ_R \subset N \times N$. Let Ω be the set of all random orderings on N. An *initial segment* in a random order $\succ_R$ is a set of the form:

$$I(s, \succ_R) = \{j \in N | s \succ_R j\}, \text{ for each } s \in N$$

We view $I(s, \succ_R)$ as the set of players who are before player s in the random order $\succ_R$. *A marginal contribution function* in an order $\succ_R$ for game V is a measure $\phi^R V$ on $(N, \mathcal{P}_N)$ that satisfies:

$$(\phi^R V)(\{i\}) = V\left(I(i, \succ_R) \cup \{i\}\right) - V\left(I(i, \succ_R)\right) \tag{3}$$

Define an operator $\Phi : G_N \to G_N$ by

$$(\Phi V)(\{i\}) = \frac{1}{n!} \sum_{\succ_R \in \Omega} (\phi^R V)(\{i\}) \tag{4}$$

It is easy to check that Φ in (4) coincides with Φ in (2).

***

For permutations of $N = \{1, 2, ..., n\}$, a *transposition* is a 2-cycle $(a\ b)$. For example $(1\ 3) = [3\ 2\ 1\ 4\ 5 ... n]$. A *simple transposition* (also called a *simple reflection*) is a transposition of two consecutive letters: $s_i \equiv (i, i+1)$. For permutation group on N, there are $n-1$ simple trnaspositions $s_2 = (1\ 2), s_3 = (2\ 3), \cdots, s_n = (n\ n+1)$ which generate the whole group of permutations through group compositions.

2.2 Reformulation of Random Order Value

In this section we reformulate the above random order approach by utilizing the group structure of the set of permutations. We prove the existence and uniqueness of semi-value and value operators on G_N by characterizing the operators in terms of invariant Haar measure on the group of symmetry. Insights that are relevant for extending our approach to games with continuum of players are stated as remarks.

Note that each $\theta \in \Theta$ generates an order, denoted as $\succ_\theta$ defined as follows:

$$\text{for } i, j \in N,\ i \succ_\theta j \quad \Leftrightarrow \quad \theta(i) > \theta(j) \tag{5}$$

It is easy to verify that $\succ_\theta$ is an order on N and that $\succ_\theta$ establishes an one-one correspondence between Θ and Ω. We define initial segments of random order $\succ_\theta$ by

$$I(s, \succ_\theta) = \{t \in N \mid \theta(t) < \theta(s)\}$$

Calculation of marginal contributions function in a random order generated by $\theta \in \Theta$ is defined as in (3) with R replaced by θ.

Let $\mathcal{B}_\Theta$ be a $\sigma-$algebra of the set of orders, Θ, and μ be a measure on $(\Theta, \mathcal{B}_\Theta)$. A measure space, $(\Theta, \mathcal{B}_\Theta, \mu)$ is said to be a set of *random orders*. For finite Θ we take the $\mathcal{B}_\Theta$ to be the power set of Θ. We define *expected marginal value* operator $\Phi_\mu : G_N \to FA$ by

$$(\Phi_\mu V)(S) = \int_\Theta (\phi^\theta V)(S) d\mu(\theta), \text{ for } V \in G_N, S \in \mathcal{P}_N \tag{6}$$

Since $\mathcal{B}_\Theta$ is the power set, $(\phi^\theta V)(S)$ is integrable for all $V \in G_N$ and $S \in \mathcal{P}_N$, and hence Φ_μ is well defined.The following lemma will be used in the proof of theorem 1 and other results.

Lemma 1. Let $S \subset \Re$, and $\theta : S \to S$, and $\pi : S \to S$ be two automorphisms of S. Denote by $I(s, \succ_\theta) = \{t \in S \mid \theta(t) < \theta(s)\}$ for an automorphism, θ. Then, $\pi^{-1}\left(I(s, \succ_\theta)\right) = I\left(\pi^{-1}(s), \succ_{\theta\pi}\right)$.

PROOF:

$$\begin{aligned} x \in \pi^{-1}\left(I(s, \succ_\theta)\right) &\Leftrightarrow \pi(x) \in I\left(s, \succ_\theta\right) \\ &\Leftrightarrow \theta(\pi(x)) < \theta(s) \\ &\Leftrightarrow (\theta\pi)(x) < (\theta\pi)\pi^{-1}(s) \\ &\Leftrightarrow x \in I\left(\pi^{-1}(s), \succ_{\theta\pi}\right) \end{aligned}$$

Q.E.D.

Proposition 1.

$$(\phi^{\theta}(\pi^* V))(S) = (\phi^{\theta\pi} V) \left(\pi^{-1}(S)\right), \forall\, S \in \mathcal{P}_N \tag{7}$$

PROOF:

$$\begin{aligned}
(\phi^{\theta}(\pi^* V))(\{i\}) &= (\pi^* V)(I(i, \succ_\theta) \cup \{i\}) - (\pi^* V)(I(i, \succ_\theta) \\
&= V(\pi^{-1}\left(I(i, \succ_\theta) \cup \{i\}\right)) - V(\pi^{-1}\left(I(i, \succ_\theta)\right)) \\
&= V\left(I(\pi^{-1}(i), \succ_{\theta\pi}) \cup \pi^{-1}(i)\right) - V\left(I(\pi^{-1}(i), \succ_{\theta\pi})\right) \text{ using lemma 1} \\
&= (\phi^{\theta\pi} V)\left(\pi^{-1}(i)\right), \forall\, i \in N
\end{aligned}$$

Hence, $(\phi^{\theta}(\pi^* V))(S) = (\phi^{\theta\pi} V)\left(\pi^{-1}(S)\right), \forall\, S \in \mathcal{P}_N$.

Q.E.D.

We now show that symmetry of Φ_μ with respect to Θ implies that the measure space $(\Theta, \mathcal{B}_\Theta, \mu)$ is a measurable group as defined below.

Definition 1. : Let Θ be a group. A measure space $(\Theta, \mathcal{B}_\Theta, \Gamma)$ is a *measurable group* if the map $(\theta_1, \theta_2) \to \theta_1\theta_2^{-1}$ from $(\Theta \times \Theta, \mathcal{B}_\Theta \times \mathcal{B}_\Theta)$ onto $(\Theta, \mathcal{B}_\Theta)$ is measurable, and Γ is $\sigma-$finite, not identically zero, and right invariant, i.e., $\mu(E\theta) = \mu(E) \forall E \in \mathcal{B}_\Theta$.

We first investigate this for the space of three player games in example 2 below. The example also illustrates many concepts that we have used so far.

Example 1. Let the set of players be denoted by $N = \{1, 2\}$. The following are the two automorphisms of N and their inverses; each representing a distinct random order, and let μ_i be the mass given by the measure μ of equation (6) to the random order, $\succ_{\theta_i}$ $i = 1$ and 2.

$$\boxed{\begin{array}{ccccccc} \theta_1 & = & [1\,2\,] & = & \theta_1^{-1} & \equiv & e \\ \theta_2 & = & [2\,1] & = & \theta_2^{-1} & & \end{array}} \quad \begin{array}{ccccccc} I(1, \succ_{\theta_1}) & = & \emptyset & ; & I(2, \succ_{\theta_1}) & = & \{1\} \\ I(1, \succ_{\theta_2}) & = & \{2\} & ; & I(2, \succ_{\theta_2}) & = & \emptyset \end{array}$$

Let μ_i be the mass at the random order generated by $\succ_{\theta_i}$, $i = 1$ and 2. Consider the game V in Box 1. For the automorphism, $\theta_2 : \begin{pmatrix} 2 & 1 \end{pmatrix}$, the transformed game, $\theta_2^* V$ as defined in Eq (6) is shown in Box 1.2:

$$\boxed{\begin{array}{lcl} \multicolumn{3}{l}{Box\ 1:} \\ V(\{1\}) & = & v_1 \\ V(\{2\}) & = & v_2 \\ V(\{1,2\}) & = & v_{12} \end{array}} \quad \boxed{\begin{array}{lcl} \multicolumn{3}{l}{Box\ 1.2:} \\ \left(\phi^{\theta_1} V\right)(\{1\}) & = & v_1 \\ \left(\phi^{\theta_1} V\right)(\{2\}) & = & v_{12} - v_1 \\ \left(\phi^{\theta_2} V\right)(\{1\}) & = & v_{12} - v_2 \\ \left(\phi^{\theta_2} V\right)(\{2\}) & = & v_2 \end{array}}$$

$$\boxed{\begin{array}{lcl} \multicolumn{3}{l}{Box\ 3:} \\ \theta_2^* V(\{1\}) & = & V\left(\theta_2^{-1}\{1\}\right) = v_2 \\ \theta_2^* V(\{2\}) & = & V\left(\theta_2^{-1}\{2\}\right) = v_1 \\ \theta_2^* V(\{1,2\}) & = & V\left(\theta_2^{-1}\{1,2\}\right) = v_{12} \end{array}} \quad \boxed{\begin{array}{lcl} \multicolumn{3}{l}{Box\ 3:} \\ \phi^{\theta_1}\left(\theta_2^* V\right)(\{1\}) & = & v_2 \\ \phi^{\theta_1}\left(\theta_2^* V\right)(\{2\}) & = & v_{12} - v_2 \\ \phi^{\theta_2}\left(\theta_2^* V\right)(\{1\}) & = & v_{12} - v_1 \\ \phi^{\theta_2}\left(\theta_2^* V\right)(\{2\}) & = & v_1 \end{array}}$$

Thus,$\Phi_\mu V$ is given by

$$(\Phi_\mu V)\begin{pmatrix} (\{1\}) \\ (\{2\}) \end{pmatrix} = \begin{pmatrix} \mu_1 v_1 + \mu_2 (v_{12} - v_2) \\ \mu_1 (v_{12} - v_1) + \mu_2 v_2 \end{pmatrix}$$

and

$$\left(\Phi_\mu \left(\theta_2^* V\right)\right)\begin{pmatrix} (\{1\}) \\ (\{2\}) \end{pmatrix} = \begin{pmatrix} \mu_1 v_2 + \mu_2 (v_{12} - v_1) \\ \mu_1 (v_{12} - v_2) + \mu_2 v_1 \end{pmatrix}$$

The following are computed using definitions:

$$\theta_2^*\left(\Phi_\mu V\right)\begin{pmatrix} (\{1\}) \\ (\{2\}) \end{pmatrix} = \begin{pmatrix} (\Phi_\mu V)\left(\theta_2^{-1}\{1\}\right) \\ (\Phi_\mu V)\left(\theta_2^{-1}\{2\}\right) \end{pmatrix} = \begin{pmatrix} \mu_1 (v_{12} - v_1) + \mu_2 v_2 \\ \mu_1 v_1 + \mu_2 (v_{12} - v_2) \end{pmatrix}$$

If $\Phi_\mu V$ is symmetric with respect to $\Theta = \{\theta_1, \theta_2\}$, we must have $\Phi_\mu\left(\theta_2^* V\right)(S) = \theta_2^*\left(\Phi_\mu V\right)(S)$. That means for $S = \{1\}$, we should have $\mu_1 v_2 + \mu_2 (v_{12} - v_2) = \mu_1 (v_{12} - v_1) + \mu_2 v_2$, i.e., $\mu_2 (v_{12} - v_2 - v_1) = \mu_1 (v_{12} - v_2 - v_1)$. Hence for non-additive games, we must have $\mu_1 = \mu_2$.

Example 2. Let the set of players be denoted by $N = \{1,2,3\}$. The following $\theta_i, i = 1,2,..6$ are the six automorphisms of N; each representing a distinct random order. For inverses θ_i^{-1} are also shown. A base for the permutation group N is the set of simple transitions θ_2, and θ_3. This is vertified in the last column. Let μ_i be the mass given by the measure μ of equation (6) to the random order, $\succ_{\theta_i}$ $i = 1, ..., 6$.

	ordering of players by $\succ_{\theta_i}$
$\theta_1 = [1\,2\,3] = \theta_1^{-1} \equiv e$	$<1\,2\,3>$
$\theta_2 = [2\,1\,3] = \theta_2^{-1}$	$<2\,1\,3>$
$\theta_3 = [1\,3\,2] = \theta_3^{-1}$	$<1\,3\,2>$
$\theta_4 = [3\,2\,1] = \theta_4^{-1} = \theta_2 \circ \theta_3 \circ \theta_2$	$<3\,2\,1>$
$\theta_5 = [2\,3\,1] = \theta_6^{-1} = \theta_2 \circ \theta_3$	$<3\,1\,2>$
$\theta_6 = [3\,1\,2] = \theta_5^{-1} = \theta_3 \circ \theta_2$	$<2\,3\,1>$

The group multiplication table in which an entry corresponding to row θ_r and column θ_c is the permutation $\theta_r \circ \theta_c$ is given below

element	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
θ_1	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
θ_2	θ_2	θ_1	θ_5	θ_6	θ_3	θ_4
θ_3	θ_3	θ_6	θ_1	θ_5	θ_4	θ_2
θ_4	θ_4	θ_5	θ_6	θ_1	θ_2	θ_3
θ_5	θ_5	θ_4	θ_2	θ_3	θ_6	θ_1
θ_6	θ_6	θ_3	θ_4	θ_2	θ_1	θ_5

Consider the game V in Box 2. For the automorphisms, θ_2 and θ_3, the transformed games, $\theta_2^* V$ and $\theta_3^* V$ are shown in Box 2.1 and Box 2.2:

$$
\begin{array}{lcl}
\textit{Box}\ 2: & & \\
V(1) & = & v_1 \\
V(2) & = & v_2 \\
V(1,2) & = & v_{12} \\
V(2,3) & = & v_{23} \\
V(1,2,3) & = & v_{123} \\
V(3) & = & v_3 \\
V(1,3) & = & v_{13}
\end{array}
\qquad
\begin{array}{lcl}
\textit{Box}\ 3: & & \\
\theta_2^* V(1) & = & v_2 \\
\theta_2^* V(2) & = & v_1 \\
\theta_2^* V(1,2) & = & v_{12} \\
\theta_2^* V(2,3) & = & v_{13} \\
\theta_2^* V(1,2,3) & = & v_{123} \\
\theta_2^* V(3) & = & v_3 \\
\theta_2^* V(1,3) & = & v_{23}
\end{array}
\qquad
\begin{array}{lcl}
\textit{Box}\ 4: & & \\
\theta_3^* V(1) & = & v_1 \\
\theta_3^* V(2) & = & v_3 \\
\theta_3^* V(1,2) & = & v_{13} \\
\theta_3^* V(2,3) & = & v_{23} \\
\theta_3^* V(1,2,3) & = & v_{123} \\
\theta_3^* V(3) & = & v_2 \\
\theta_3^* V(1,3) & = & v_{12}
\end{array}
$$

The following boxes show the marginal contributions for the the game V in Box 2,

$$
\begin{array}{lcl}
\textit{Box}\ 2.1 & & \\
\left(\phi^{\theta_1} V\right)(1) & = & v_1 \\
\left(\phi^{\theta_2} V\right)(1) & = & v_{12} - v_2 \\
\left(\phi^{\theta_3} V\right)(1) & = & v_1 \\
\left(\phi^{\theta_4} V\right)(1) & = & v_{123} - v_{23} \\
\left(\phi^{\theta_5} V\right)(1) & = & v_{13} - v_3 \\
\left(\phi^{\theta_6} V\right)(1) & = & v_{123} - v_{23}
\end{array}
\qquad
\begin{array}{lcl}
\textit{Box}\ 2.2 & & \\
\left(\phi^{\theta_1} V\right)(2) & = & v_{12} - v_1 \\
\left(\phi^{\theta_2} V\right)(2) & = & v_2 \\
\left(\phi^{\theta_3} V\right)(2) & = & v_{123} - v_{13} \\
\left(\phi^{\theta_4} V\right)(2) & = & v_{23} - v_3 \\
\left(\phi^{\theta_5} V\right)(2) & = & v_{123} - v_{13} \\
\left(\phi^{\theta_6} V\right)(2) & = & v_2
\end{array}
\qquad
\begin{array}{lcl}
\textit{Box}\ 2.3 & & \\
\left(\phi^{\theta_1} V\right)(3) & = & v_{123} - v_{12} \\
\left(\phi^{\theta_2} V\right)(3) & = & v_{123} - v_{12} \\
\left(\phi^{\theta_3} V\right)(3) & = & v_{13} - v_1 \\
\left(\phi^{\theta_4} V\right)(3) & = & v_3 \\
\left(\phi^{\theta_5} V\right)(3) & = & v_3 \\
\left(\phi^{\theta_6} V\right)(3) & = & v_{23} - v_2
\end{array}
$$

The following boxes show the marginal contributions for the the game $\theta_2^* V$ in Box 3,

$$\begin{array}{lll}
\textit{Box } 3.1 & \textit{Box } 3.2 & \textit{Box } 3.3 \\
\left(\phi^{\theta_1}\theta_2^*V\right)(1) = v_2 & \left(\phi^{\theta_1}\theta_2^*V\right)(2) = v_{12} - v_2 & \left(\phi^{\theta_1}\theta_2^*V\right)(3) = v_{123} - v_{12} \\
\left(\phi^{\theta_2}\theta_2^*V\right)(1) = v_{12} - v_1 & \left(\phi^{\theta_2}\theta_2^*V\right)(2) = v_1 & \left(\phi^{\theta_2}\theta_2^*V\right)(3) = v_{123} - v_{12} \\
\left(\phi^{\theta_3}\theta_2^*V\right)(1) = v_2 & \left(\phi^{\theta_3}\theta_2^*V\right)(2) = v_{123} - v_{23} & \left(\phi^{\theta_3}\theta_2^*V\right)(3) = v_{13} - v_1 \\
\left(\phi^{\theta_4}\theta_2^*V\right)(1) = v_{123} - v_{13} & \left(\phi^{\theta_4}\theta_2^*V\right)(2) = v_{13} - v_3 & \left(\phi^{\theta_4}\theta_2^*V\right)(3) = v_3 \\
\left(\phi^{\theta_5}\theta_2^*V\right)(1) = v_{23} - v_3 & \left(\phi^{\theta_5}\theta_2^*V\right)(2) = v_{123} - v_{23} & \left(\phi^{\theta_5}\theta_2^*V\right)(3) = v_3 \\
\left(\phi^{\theta_6}\theta_2^*V\right)(1) = v_{123} - v_{13} & \left(\phi^{\theta_6}\theta_2^*V\right)(2) = v_1 & \left(\phi^{\theta_6}\theta_2^*V\right)(3) = v_{13} - v_1
\end{array}$$

The following boxes show the marginal contributions for the the game $\theta_3^* V$ in Box 4,

$$\begin{array}{lll}
\textit{Box } 4.1 & \textit{Box } 4.2 & \textit{Box } 4.3 \\
\left(\phi^{\theta_1}\theta_3^*V\right)(1) = v_1 & \left(\phi^{\theta_1}\theta_3^*V\right)(2) = v_{13} - v_1 & \left(\phi^{\theta_1}\theta_3^*V\right)(3) = v_{123} - v_{13} \\
\left(\phi^{\theta_2}\theta_3^*V\right)(1) = v_{13} - v_3 & \left(\phi^{\theta_2}\theta_3^*V\right)(2) = v_3 & \left(\phi^{\theta_2}\theta_3^*V\right)(3) = v_{123} - v_{13} \\
\left(\phi^{\theta_3}\theta_3^*V\right)(1) = v_1 & \left(\phi^{\theta_3}\theta_3^*V\right)(2) = v_{123} - v_{12} & \left(\phi^{\theta_3}\theta_3^*V\right)(3) = v_{23} - v_2 \\
\left(\phi^{\theta_4}\theta_3^*V\right)(1) = v_{123} - v_{23} & \left(\phi^{\theta_4}\theta_3^*V\right)(2) = v_{23} - v_2 & \left(\phi^{\theta_4}\theta_3^*V\right)(3) = v_2 \\
\left(\phi^{\theta_5}\theta_3^*V\right)(1) = v_{12} - v_2 & \left(\phi^{\theta_5}\theta_3^*V\right)(2) = v_{123} - v_{12} & \left(\phi^{\theta_5}\theta_3^*V\right)(3) = v_2 \\
\left(\phi^{\theta_6}\theta_3^*V\right)(1) = v_{123} - v_{23} & \left(\phi^{\theta_6}\theta_3^*V\right)(2) = v_3 & \left(\phi^{\theta_6}\theta_3^*V\right)(3) = v_{23} - v_3
\end{array}$$

The following are computed using definitions:

$$(\Phi_\mu V)\begin{pmatrix}(1)\\(2)\\(3)\end{pmatrix} = \begin{pmatrix} (\mu_1+\mu_3)v_1 + \mu_2(v_{12}-v_2) + (\mu_4+\mu_6)(v_{123}-v_{23}) + \mu_5(v_{13}-v_3) \\ \mu_1(v_{12}-v_1) + (\mu_2+\mu_6)v_2 + (\mu_3+\mu_5)(v_{123}-v_{13}) + \mu_4(v_{23}-v_3) \\ (\mu_1+\mu_2)(v_{123}-v_{12}) + \mu_3(v_{13}-v_1) + (\mu_4+\mu_5)v_3 + \mu_6(v_{23}-v_2) \end{pmatrix} \tag{8}$$

I follow two steps: Step 1: Assume that the result is true for games with $n = 2$, and we will show that the result is also true for $n = 3$ players.Assume that player 3 is a null player, i.e., $v_{123} = v_{12} + v_3$, $v_{23} = v_2 + v_3$, $v_{13} = v_1 + v_3$.

Let us note that expected marginal value of players 1 and 2 are given by:

$$(\Phi_\mu V)\begin{pmatrix}(1)\\(2)\\(3)\end{pmatrix} = \begin{pmatrix} (\mu_1+\mu_3+\mu_5)v_1 + (\mu_2+\mu_4+\mu_6)(v_{12}-v_2) \\ (\mu_1+\mu_3+\mu_5)(v_{12}-v_1) + (\mu_2+\mu_4+\mu_6)v_2 \\ (\mu_1+\mu_2+\mu_3+\mu_4+\mu_5+\mu_6)v_3 \end{pmatrix} \tag{9}$$

Note that the above game is essentially a game of two players, $\{1, 2\}$. Denote the two random orders $\{1, 2\}$ by [1 2] and [2 1]. Let us identify these two random orders of two players respectively with the equivalence class of random orders $\{\theta_1, \theta_3, \theta_5\}$ and

$\{\theta_2, \theta_4, \theta_6\}$, of three players, $\{1, 2, 3\}$ where two random orders are equivalent if by deleting the null player, 3, we are left with the same ordering of the players $\{1, 2\}$. Since player 3 is dummy all random orders belonging to an equivalence class assign the same marginal contribution to the players 1 and 2 as in the corresponding random orders of the two players in the two players game. Let μ'_1 and μ'_2 be the point masses of random orders $(1\ 2)$ and $(2\ 1)$ in the two player representation of the same game. Thus we have

$$(\Phi_{\mu'} V)\begin{pmatrix}(1)\\(2)\end{pmatrix} = \begin{pmatrix}\mu'_1 v_1 + \mu'_2(v_{12} - v_2)\\ \mu'_1(v_{12} - v_1) + \mu'_2 v_2\end{pmatrix}$$

Since the result is assumed to be true for games with 2 players, we have that $\mu'_1 = \mu'_2$. Thus for the equivalent three person game, we must have

$$\mu_1 + \mu_3 + \mu_5 = \mu_2 + \mu_4 + \mu_6 \tag{10}$$

Treating players 2 and 1 as null players in turn and proceeding as above, we also have the following constraints on the $\mu'_j s$:

$$\mu_1 + \mu_2 + \mu_3 = \mu_4 + \mu_5 + \mu_6 \tag{11}$$

$$\mu_1 + \mu_2 + \mu_6 = \mu_3 + \mu_4 + \mu_5 \tag{12}$$

The above system of constraints (10)-(12) and the constraint that $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 = 1$ provide 4 equations for six unknowns $\mu_1, \mu_2, ..., \mu_6$. I derive two more equations using the simple game with $R = \{1, 2, 3\}$ in the definition (1) and the symmetry condition for this game $\theta^*(\Phi_\mu V)(S) = \Phi_\mu(\theta^* V)(S)$ for two basis permutations $\theta = \theta_2$ and $\theta = \theta_3$. To that end we use the definition to calculate

$$\Phi_\mu(\theta_2^* V)\begin{pmatrix}(1)\\(2)\\(3)\end{pmatrix} = \begin{pmatrix}(\mu_1 + \mu_3)v_2 + \mu_2(v_{12} - v_1) + (\mu_4 + \mu_6)(v_{123} - v_{13}) + \mu_5(v_{23} - v_3)\\ \mu_1(v_{12} - v_2) + (\mu_2 + \mu_6)v_1 + (\mu_3 + \mu_5)(v_{123} - v_{23}) + \mu_4(v_{13} - v_3)\\ (\mu_1 + \mu_2)(v_{123} - v_{12}) + \mu_3(v_{23} - v_2) + (\mu_4 + \mu_5)\, v_3 + \mu_6\,(v_{13} - v_1)\end{pmatrix}$$

and,

$$\Phi_\mu(\theta_3^* V)\begin{pmatrix}(1)\\(2)\\(3)\end{pmatrix} = \begin{pmatrix}(\mu_1 + \mu_3)v_1 + \mu_2(v_{13} - v_3) + (\mu_4 + \mu_6)(v_{123} - v_{23}) + \mu_5(v_{12} - v_2)\\ \mu_1(v_{13} - v_1) + (\mu_2 + \mu_6)v_3 + (\mu_3 + \mu_5)(v_{123} - v_{12}) + \mu_4(v_{23} - v_2)\\ (\mu_1 + \mu_2)(v_{123} - v_{13}) + \mu_3(v_{12} - v_1) + (\mu_4 + \mu_5)\, v_2 + \mu_6\,(v_{23} - v_3)\end{pmatrix}$$

Symmetric of Φ_μ for all games implies $\theta_2^*(\Phi_\mu V)\,(1) = \Phi_\mu(\theta_2^* V)(1)$, but $\theta_2^*(\Phi_\mu V)\,(1) = (\Phi_\mu V)\left(\theta_2^{-1}(1)\right) = (\Phi_\mu V)\,(2)$. Thus,we should have $(\Phi_\mu V)\,(2) = \Phi_\mu(\theta_2^* V)(1)$.Which for the simple game with $R = \{1,2,3\}$ in the definition (1) i.e., $v_{123} = 1$, and all other v's are $= 0$ requires that

$$\mu_3 + \mu_5 = \mu_4 + \mu_6 \tag{13}$$

Carrying out the above steps for the above simple game and symmetry with respect to θ_3 and equating $\theta_3^*(\Phi_\mu V)\,(3) = \Phi_\mu(\theta_2^* V)(3)$, i.e., $(\Phi_\mu V)\,(2) = \Phi_\mu(\theta_3^* V)(3)$ will produce

$$\mu_3 + \mu_5 = \mu_1 + \mu_2 \tag{14}$$

Note that (10)-(12) imply $\mu_1 = \mu_4, \mu_2 = \mu_5$, and $\mu_3 = \mu_6$. Using these in (13) and (14) yiels $\mu_1 = \mu_5$ and $\mu_1 = \mu_3$. Thus $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6$. Hence the claim is established.

2.2.1 Another Look at Example 2

Two steps of Example 2 could be done the following way: **Step 1**: Assume that the result is true for games with $n = 2$, and we will show that the result is also true for $n = 3$ players.Assume that player 3 is a null player, i.e., $v_{123} = v_{12} + v_3$, $v_{23} = v_2 + v_3$, $v_{13} = v_1 + v_3$.

It follows from (8) that

$$(\Phi_\mu V)\begin{pmatrix}(1)\\(2)\end{pmatrix} = \begin{pmatrix}(\mu_1+\mu_3+\mu_5)v_1 + (\mu_2+\mu_6+\mu_4)(v_{12}-v_2)\\(\mu_1+\mu_3+\mu_5)(v_{12}-v_1) + (\mu_2+\mu_6+\mu_4)\,v_2\end{pmatrix} \tag{15}$$

Notice that disregarding the player 3, the value formula coinsides with the value formula of a 2 player game.

$$(\Phi_\mu V)\begin{pmatrix}(1)\\(2)\end{pmatrix} = \begin{pmatrix}\tilde{\mu}_1 v_1 + \tilde{\mu}_2(v_{12}-v_2)\\\tilde{\mu}_1(v_{12}-v_1) + \tilde{\mu}_2 v_2\end{pmatrix} \tag{16}$$

Step 2: We have seen that for two players game symmetry of the Shapley value implies $\tilde{\mu}_1 = \tilde{\mu}_2$. This also follows from the symmetry assumption of the three player game and applying the symmetry with respect θ_2, i.e.,

$$(\Phi_\mu \theta_2^* V)\begin{pmatrix}(1)\\(2)\end{pmatrix} = \begin{pmatrix}(\mu_1+\mu_3+\mu_5)v_2 + (\mu_2+\mu_6+\mu_4)\,(v_{12}-v_1)\\(\mu_1+\mu_3+\mu_5)(v_{12}-v_2) + (\mu_2+\mu_6+\mu_4)v_1\end{pmatrix} \tag{17}$$

Thus we have

$$\begin{aligned} \mu_{\theta_1} + \mu_{\theta_3} + \mu_{\theta_5} &= \mu_{\theta_2} + \mu_{\theta_6} + \mu_{\theta_4} \\ &= \mu_{\theta_1 \circ \theta_2} + \mu_{\theta_3 \circ \theta_2} + \mu_{\theta_5 \circ \theta_2} \end{aligned} \tag{18}$$

In the above the last step shows that the since the equivalent class of orders $[2\ 1]$ in the two player game is obtained by composing the simple permutation θ_2 of players 1 and 2 in the natural order θ_1. We must get the equality of probability masses for two players when we treat player 2 and player 1 as dummy player as we have seen before. For instance the simple permutation $\theta_2 = (2\ 3)$ ...

Applying θ_3 on (18), we have

$$\mu_{\theta_3} + \mu_{\theta_1} + \mu_{\theta_2} = \mu_{\theta_5} + \mu_{\theta_4} + \mu_{\theta_6} \tag{19}$$

From (18) and (19) we get

$$\mu_{\theta_2} = \mu_{\theta_5} \tag{20}$$

Applying θ_4 on (20), we have

$$\mu_{\theta_6} = \mu_{\theta_3} \tag{21}$$

We have seen that Symmetric of Φ_μ for all games implies $\theta_2^*(\Phi_\mu V)(1) = \Phi_\mu(\theta_2^* V)(1)$, but $\theta_2^*(\Phi_\mu V)(1) = (\Phi_\mu V)\left(\theta_2^{-1}(1)\right) = (\Phi_\mu V)(2)$. Thus,we should have $(\Phi_\mu V)(2) = \Phi_\mu(\theta_2^* V)(1)$. Which for the simple game with $R = \{1, 2, 3\}$ in the definition (1) i.e., $v_{123} = 1$, and all other v's are $= 0$ requires that

$$\mu_{\theta_3} + \mu_{\theta_5} = \mu_{\theta_6} + \mu_{\theta_4} \tag{22}$$

From (18) and (22) it follows that

$$\mu_{\theta_1} = \mu_{\theta_2} \tag{23}$$

Applying θ_3 on (18) and (**??**) or equivalently on (21), and noticing from the multiplication table that $\theta_1 \circ \theta_3 = \theta_3$ and $\theta_2 \circ \theta_3 = \theta_5$,we get

$$\mu_{\theta_3} = \mu_{\theta_5} \tag{24}$$

Similarly, applying θ_3 on (18) and (**??**) or equivalently on (21) we get,

$$\mu_{\theta_4} = \mu_{\theta_6} \tag{25}$$

Combining the above, we get $\mu_{\theta_1} = \mu_{\theta_2} = \mu_{\theta_3} = \mu_{\theta_4} = \mu_{\theta_5} = \mu_{\theta_6}$ as a consequence of the symmytry axiom to hold on all games.

Remark 1. The result in the above example that symmetry of Φ_μ implies right invariance of μ could be proved for more general games inductively on the number of players: Assuming that the result is true for games with n players, we can show that the result is also true for $n+1$ players. Which is proved in the theorem 1

The converse of the above result is also true, as stated and proved in the following theorem:

Theorem 1. The expected marginal value operator Φ_μ on G_N is symmetric with respect to Θ if and only if $(\Theta, \mathcal{B}_\Theta, \mu)$ is a measurable group.

PROOF of Theorem 1: We have already established the only if part. Let us prove the if part here. Let μ an invariant measure on $(\Theta, \mathcal{B}_\Theta)$. We want to show that Φ_μ is symmetric with respect to Θ. Let $\pi \in \Theta$ and $V \in G_N$ be arbitrarily fixed, we want to show that $\Phi_\mu \pi^* = \pi^* \Phi_\mu$. Note that

$$\begin{aligned}
(\phi^\theta(\pi^* V))(\{i\}) &= (\pi^* V)(I(i, \succ_\theta) \cup \{i\}) - (\pi^* V)(I(i, \succ_\theta)) \\
&= V(\pi^{-1}(I(i, \succ_\theta) \cup \{i\})) - V(\pi^{-1}(I(i, \succ_\theta))) \\
&= V\left(I(\pi^{-1}(i), \succ_{\theta\pi}) \cup \pi^{-1}(i)\right) - V\left(I(\pi^{-1}(i), \succ_{\theta\pi})\right) \text{ using lemma 1} \\
&= (\phi^{\theta\pi} V)\left(\pi^{-1}(i)\right), \ \forall\, i \in N
\end{aligned}$$

Hence,

$$(\phi^\theta(\pi^* V))(S) = (\phi^{\theta\pi} V)\left(\pi^{-1}(S)\right), \ \forall\, S \in \mathcal{P}_N \tag{26}$$

Hence,

$$\begin{aligned}
\Phi_\mu(\pi^* V)(S) &= \int_\Theta (\phi^\theta(\pi^* V)(S) d\mu(\theta) \\
&= \int_\Theta (\phi^{\theta\pi} V)\left(\pi^{-1}(S)\right) d\mu(\theta), \quad \text{(by equation 26)} \\
&= \int_\Theta (\phi^{\theta\pi} V)\left(\pi^{-1}(S)\right) d\mu(\theta\pi), \quad \text{(since } \mu \text{ is right invariant)} \\
&= (\Phi_\mu V)\left(\pi^{-1}(S)\right) \\
&= \pi^*(\Phi_\mu V)(S), \ \forall\, S \in \mathcal{P}_N, \ V \in G_N
\end{aligned}$$

Thus, $\Phi_\mu \pi^* = \pi^* \Phi_\mu$.

Q.E.D.

Remark 2. Is there a link between invariance of Φ_μ and the invariant integration on compact groups? To find the link, notice that symmetry of Φ_μ with respect to Θ implies that for all $\pi \in \Theta$, we have

$$\pi^* \left(\Phi_\mu V\right)(S) = \left(\Phi_\mu(\pi^* V)\right)(S) \tag{27}$$

Consider the left hand side of (27):

$$\begin{aligned}
\pi^* \left(\Phi_\mu V\right)(S) &= \left(\Phi_\mu V\right)\left(\pi^{-1}(S)\right) \text{ by definition of } \pi^* \\
&= \int_\Theta \left(\phi^{\hat\theta} V\right)(\pi^{-1}(S))d\mu(\hat\theta) \text{ by definition of } \Phi_\mu \\
&= \int_\Theta \left(\phi^{\theta\pi} V\right)(\pi^{-1}(S))d\mu(\theta\pi), \text{ for some } \theta \in \Theta, \ldots\ldots. \text{ (A)} \\
&= \int_\Theta \left(\phi^{\theta}(\pi^* V)\right)(S)d\mu(\theta\pi), \text{ by equation (26)} \\
&= \int_\Theta \left(\phi^{\theta\pi^{-1}}(\pi^* V)\right)(S)d\mu(\theta)
\end{aligned}$$

In establishing the equality (A) above we have used the fact that since Θ is a group, $\exists\theta \in \Theta$ such that $\hat\theta = \theta\pi$; we have used the translation map $\theta \mapsto \theta\pi^{-1}$ and the change of variable formula on the previous step in establishing the last equality.

Applying definition of Φ_μ on $\pi^* V$ on the right hand side of (27), and denoting $h(\theta) \equiv \phi^\theta(\pi^* V)$ we have

$$\int_\Theta h(\theta\pi^{-1})d\mu(\theta) = \int_\Theta h(\theta)d\mu(\theta), \ \forall\pi \in \Theta \tag{28}$$

Note that $h(\theta) \in L_1(\Theta, \mu)$ in (28) varies with $V \in G_N$ and $S \in \mathcal{P}_N$. If we could assure that $\mathcal{F} = \{h(\theta) | V \in G_N, S \in \mathcal{P}_N\}$ is dense in $L_1(\Theta, \mu)$, we know from the theory of integration on compact groups that μ is necessarily invariant. This might be the case, but we have not tried this.

It is easy to check that Φ_μ defined in (6) is linear on G_N. Because of theorem 1, Φ_μ defined in (6) with μ as a right invariant measure on $(\Theta, \mathcal{B}_\Theta)$, is said to be *a random order semi-value operator* on G_N; and an efficient random order semi-value operator on G_N is said to be a *random order value operator on* G_N. The following theorem assures the existence and uniqueness of random order semi-value and value operator on G_N.

Theorem 2. There exists a random order semi-value operator Φ_μ on G_N. Moreover, a random order semi-value operator is unique up to multiplication by a constant, i.e., if Φ_μ

and $\Phi_{\mu'}$ are two random order semi-value operators, then there exists $c \in \Re$ such that $\Phi_\mu = c\Phi_{\mu'}$.

The following concept will be used in the proof of the above theorem.

Definition 2. : A group Θ with a topology $\mathcal{T}$ is a topological group if $(\Theta, \mathcal{T})$ is a Hausdorff topological space and the map $(\theta_1, \theta_2) \to \theta_1\theta_2^{-1}$ from $(\Theta \times \Theta, \mathcal{T} \times \mathcal{T})$ onto $(\Theta, \mathcal{T})$ is continuous, where $\mathcal{T} \times \mathcal{T}$ denotes the product topology.

PROOF OF THEOREM 2: Let us equip Θ with the discrete topology so that Θ is a locally compact topological group. Let $\mathcal{B}_\Theta$ be the Borel σ-algebra of Θ. By Haar measure theorem (see, Halmos, 1950, Theorem B, p.254 or Parthasarathy, 1977, Proposition 54.2 there exists a regular Borel measure on $(\Theta, \mathcal{B}_\Theta)$ which is right invariant and such a measure is unique up to a scalar multiplication. Let μ be such a measure on $(\Theta, \mathcal{B}_\Theta)$. The uniqueness up to a scalar multiplication of Φ_μ follows from the uniqueness up to a scalar multiplication of μ.

Q.E.D.

Remark 3. The existence of a right invariant measure on $(\Theta, \mathcal{B}_\Theta)$ and its uniqueness up to scalar multiplication in the above proof can be shown directly noting that Θ is finite. However, the Haar measure theorem is true for any locally compact topological group and we will use this result later for games with continuum of players. Although we use only right invariance of a measure on a group, there is a parallel concept of left invariance. In general they are not the same. One can, however, construct one type of invariant measure from the other kind. If a right invariant measure is totally finite, then two notions of invariance coincide.

Corollary 1. *There exists a unique random order value operator on G_N.*

PROOF: It is easy to note that Φ_μ is efficient $\Leftrightarrow$ μ is a probability measure on $(\Theta, \mathcal{B}_\Theta)$. Note that since the identity map $i : N \to N$ is in Θ, right invariance of $\mu \Leftrightarrow \mu(i) = \mu(i \circ \theta) = \mu(\theta)$, for all $\theta \in \Theta$. But since Θ is finite, there exists only one right invariant probability measure μ on $(\Theta, \mathcal{B}_\Theta)$ that gives equal weight to each element of Θ. Hence there exists a unique random order value operator Φ_μ on G_N.

Q.E.D.

Remark 4. It is clear from theorem 2 and its corollary that given the measure structure $\mathcal{B}_\Theta$ on Θ, the domain of a semi-value operator Φ_μ is independent of the particular choice of μ and there is only one value operator. In the following proposition we show that while a coarser measure structure restricts the domain of the random order semi-value operator, but the random order semi-value and value is independent of a particular choice of measure structure for games in the common domain.

Proposition 2. Let $\left(\Theta, \mathcal{B}'_\Theta, \mu'\right)$ be another measurable group such that $\mathcal{B}_\Theta \subset \mathcal{B}'_\Theta$. Let $Q \subset G_N$ be the set of games on which $\Phi_{\mu'}$ in (6) is well defined. Then $\exists c \in \Re$ such that $\Phi_\mu = c\Phi_{\mu'}$ on Q. If Φ_μ and $\Phi_{\mu'}$ are random order value operators then $\Phi_\mu = \Phi_{\mu'}$ on Q.

PROOF: Note that μ restricted to $\mathcal{B}'_\Theta$ is also a right invariant measure on $\left(\Theta, \mathcal{B}'_\Theta\right)$. Hence by Haar measure theorem, there exists $c \in \Re$ such that $\mu = c\mu'$ on $\mathcal{B}'_\Theta$. Hence, $\Phi_\mu = c\Phi_{\mu'}$ on Q. If Φ_μ and $\Phi_{\mu'}$ are random order value operators, then μ and μ' are probability measures. Hence, $c = 1$.

Q.E.D.

Remark 5. It is clear that if we restrict the symmetry of Φ_μ in (6) to a proper subgroup, Θ' of the group of all automorphisms, Θ, all the results about random order semi-value and Shapley value operator hold with respect to this subgroup of automorphisms. While the domain of the random order semi-value and value operator with respect to Θ' is G_N, the operator need not be symmetric with respect to the full group of automorphisms. However, it will be the case on a subspace of games,

$$Q_{\Theta'} = \left\{V \in G_N \middle| \text{ for each } \theta \in \Theta, \exists \theta' \in \Theta' \text{ such that } \phi^\theta V = \phi^{\theta'} V\right\}$$

For instance, in example 2, suppose we restrict the symmetry of Φ_μ to the subgroup $\Theta' = \left\{\theta_1^{-1}, \theta_3^{-1}\right\}$. Let $Q_{\Theta'}$ be the set of all games in example 2 such that $r_7 = r_2 + r_6$, $r_4 = r_3 + r_6$, and $r_5 = r_1 + r_6$, and $r_j \in \Re\ \forall j$. Or in other words the space of all three person games in which player 3 is a null player. It is trivial to check that $Q_{\Theta'}$ is symmetric with respect to Θ' and that random order value (semi-value) of a game in $Q_{\Theta'}$ with respect to the random orders generated by the proper subgroup Θ' of automorphisms coincides with the random order value (semi-value) with respect to the full group of automorphisms.

3 Random order value for games with a continuum of players

We use the above reformulation to explore the possibility of constructing random order semi-value operators for games with continuum of players.

Let $I = [0,1] \subset \Re$ be the set of players. Let $\mathcal{B}_I$ be the Borel sigma-algebra of I, i.e., the sigma algebra generated by the set of open intervals in I. The elements of $\mathcal{B}_I$ are the set of all possible coalitions. A *game* is a set function $V : \mathcal{B}_I \to \Re$ such that $V(\emptyset) = 0$. It is easy to verify that G_I is a linear vector space. A *null coalition of a game* V is a $T \in \mathcal{B}_I$ such that $V(S) = V(T^c \cap S), \forall S \in \mathcal{B}_I$. An *atom* of V is a coalition S such that for every coalition $T \subset S$, either T or $S \cap T^c$ is null. V *is non-atomic* if V has no atom. We denote the set of all games by G_I. By a *measure* we will mean a countably additive sign measure on $(I, \mathcal{B}_I)$. Note that a measure is also a game. A game V is *monotonic* if $S, T \in \mathcal{B}_I$, and $S \supset T \Rightarrow V(S) \geq V(T)$.

Although in our analysis of random order value we do not use any topological structure on the space of games, to facilitate comparison with Aumann and Shapley the following topological concepts are reproduced from Aumann and Shapley results. A game V is *of bounded variation* if there exist monotonic games U and W such that $V = U - W$. Let us denote the set of all non-atomic games of bounded variations by BV. It can be shown that BV is a linear space over $\Re$. Define a map $\| \,.\, \|$: BV $\to \Re$ by

$$\| V \| = inf\, \{U(I) - W(I) \mid V = U - W,\ U \text{ and } W \text{ are monotonic games}\}$$

for each $V \in$ BV. It can be shown that $\| \,.\, \|$ is a well defined norm on BV and with this norm BV is a Banach space (see **)**. A *Borel automorphism* is a measurable map θ : $(I, \mathcal{B}_I) \to (I, \mathcal{B}_I)$ such that it is one-one, onto and θ^{-1} is also measurable.

Let

$$\begin{aligned}
\text{FA} &= \text{set of finitely additive set functions} \\
\text{NA} &= \text{set of non-atomic measures on } (I, \mathcal{B}_I) \\
\text{pNA} &= \|.\| - \text{closure of linear space spanned by powers of } \mu \in \text{NA} \\
\mathcal{G} &= \text{set of all Borel automorphisms on } (I, \mathcal{B}_I).
\end{aligned}$$

It can be shown that FA, and NA and pNA are all closed subspaces of BV. The games bv$'$NA is the $\|.\| -$ closure of the space of measure valued games $(f \circ \mu)(S)$ such that μ is a nonnegative, nonatomic probability measure and $f : I \to \Re$ is a function of bounded variation, continuous at 0 and 1, and $f(0) = 0$.

An *order generated by* $\theta \in \mathcal{G}$ is an order $\succ_\theta \subset I \times I$ defined by

$$\text{for any } s, t \in I, \quad s \succ_\theta t \;\Leftrightarrow\; \theta(s) > \theta(t)$$

It can be shown that $\succ_\theta$ is a transitive, irreflexive and complete order on I. Let $\bar{I} = I \cup \{\infty\}$, and assume that $\theta(\infty) = \infty$, for all $\theta \in \mathcal{G}$. For $\theta \in \mathcal{G}$, and $s \in \bar{I}$, define *an initial segment* $I(s, \succ_\theta)$ by $I(s, \succ_\theta) = \{t \in I \mid \theta(s) > \theta(t)\}$. We view $I(s, \succ_\theta)$ as the set of players who are before player s in the random order $\succ_\theta$, $\theta \in \mathcal{G}$.

Remark 6. Aumann and Shapley, 1974[][pp.94-95]defined a transitive, irreflexive and complete order $\mathcal{R}$ on I to be *measurable* if the σ-algebra generated by the set of initial segments $\{I(s, \mathcal{R}) \mid s \in \bar{I}\}$ coincides with $\mathcal{B}_I$. It is easy to see that $\succ_\theta$ generated by a Borel automorphism $\theta \in \mathcal{G}$ is measurable in the Aumann-Shapley sense. But not every order measurable in the sense of Aumann and Shapley can be represented by a Borel automorphism. Using the insight from games with finite set of players, we note that for random order semi-value and Shapley value, we can confine the set of orders to the ones generated by the group of Borel automorphisms, $\mathcal{G}$.

Marginal contribution function of a game V *in an order* $\succ_\theta$, $\theta \in \mathcal{G}$ is a measure $(\phi^\theta V)$ on $(I, \mathcal{B}_I)$ such that

$$(\phi^\theta V)(I(s, \succ_\theta)) = V(I(s, \succ_\theta)),\ \forall\, s \in I \tag{29}$$

Proposition 3. For a game V in G_I and an order $\succ_\theta$, $\theta \in \mathcal{G}$, if a marginal contribution function $\phi^\theta V$ exists then it is unique.

PROOF: Let us denote by $[s, t)_\theta = \{j \in I \mid \theta(s) \le \theta(j) < \theta(t)\}$. Denote by $\mathcal{D}_\theta = \{[s, t)_\theta \mid s \in I, t \in \bar{I}\}$. One can easily verify that $\mathcal{D}_\theta$ is the smallest Boolean semi algebra containing all initial segments $\mathcal{I}_\theta = \{I(s, \succ_\theta) \mid s \in \bar{I}\}$. There is a unique extension of $\phi^\theta V$ from $\mathcal{I}_\theta$ to $\mathcal{D}_\theta$ such that $\phi^\theta V$ is finitely additive on $\mathcal{D}_\theta$ and equation (29) is satisfied. More precisely, note that for the initial segments in $\mathcal{D}_\theta$, equation (29) defines $\phi^\theta V$, and for all other sets in $\mathcal{D}_\theta$, there is only one way $\phi^\theta V$ can be defined as follows:

$$(\phi^\theta V)\left([s, t)_\theta\right) = V\left(I(t, \succ_\theta)\right) - V\left(I(s, \succ_\theta)\right) \text{ for } s \in I, t \in \bar{I}.$$

By corollary 16.9 in Parthasarathy [1977], there exists a unique extension of $\phi^\theta V$ to $\mathcal{B}_I$.

Q.E.D.

The following shows that (26) is true for the continuum case.

Proposition 4. Let $V \in G_I$ be such that marginal contribution function $\phi^\theta V$ exists for all $\theta \in \mathcal{G}$. Then for any $\pi, \theta \in \mathcal{G}$, we have

$$\left(\phi^\theta(\pi^* V)\right)(S) = \left(\phi^{\theta\pi} V\right)\left(\pi^{-1}(S)\right), \; \forall S \in \mathcal{B}_I \tag{30}$$

PROOF: Follows easily.

In order to adopt our random order approach for games with finite set of players, we need to extend many basic facts that were used in the finite case.

Our aim is to generate a reasonable group of random orders from $\mathcal{G}$ and equip it with a measure structure such that expected marginal contribution function gives us random order semi-value and Shapley value for a large class of games.

To that end first of all note that, in the finite player case, each $\theta \in \mathcal{G}$ generates a distinct order of the players. However, this is not the case for continuum of players. For instance, let us denote by $e \in \mathcal{G}$ the identity map of I. e generates the standard order $<$ on I. Let $\theta : I \to I$ be a (strictly increasing,) homeomorphism such that $\theta(0) = 0$ and $\theta(1) = 1$, then it can be easily verified that θ and e generate the same order.

To create a maximal group of orders from $\mathcal{G}$, one would naturally like to follow the factorization techniques of group theory as follows: Characterize the set of random orders on $I \times I$ that are induced by $\mathcal{G}$ as follows: Let us begin by defining an equivalent relation $\sim$ on $\mathcal{G} \times \mathcal{G}$ by

$$\theta_1 \sim \theta_2, \text{ for } \theta_1, \theta_2 \in \mathcal{G} \;\Leftrightarrow\; \theta_1 \text{ and } \theta_2 \text{ generate the same order on } I.$$

Denote by $\mathcal{G}_e^*$ the set of Borel automorphisms that generate the standard order of I, i.e.,

$$\mathcal{G}_e^* = \{\theta \in \mathcal{G} \mid \theta \sim e\}$$

It can be easily shown that $\mathcal{G}_e^*$ is a subgroup of $\mathcal{G}$.

For $A \subset \mathcal{G}$, and $\theta \in \mathcal{G}$, let $A\theta \equiv \{\alpha \circ \theta \mid \alpha \in A\}$. Let us denote by $\tilde{\theta} = \mathcal{G}_e^*\theta$, $\theta \in \mathcal{G}$, a right coset of the subgroup $\mathcal{G}_e^*$ generated by θ. Let $\tilde{\Theta} = \left\{\tilde{\theta} \equiv \mathcal{G}_e^*\theta \mid \theta \in \Theta\right\}$, and define the multiplication operation on the set of right cosets by

$$\tilde{\theta}_1\tilde{\theta}_2 = \left\{\theta_1 \circ \theta_2 \mid \theta_1 \in \tilde{\theta}_1, \theta_2 \in \tilde{\theta}_2\right\}$$

That is, we take $\tilde{\Theta} = \mathcal{G}/\mathcal{G}_e^*$. It is known that the set of right cosets, $\tilde{\Theta}$, with $\mathcal{G}_e$ as the identity element, $\tilde{\theta}^{-1} \equiv \left\{\theta^{-1} \mid \theta \in \tilde{\theta}\right\}$ as the inverse of $\tilde{\theta}$, and the above multiplication operation is a group if and only if $\mathcal{G}_e^*$ is a normal subgroup of $\mathcal{G}$, ($\mathcal{G}_e^*$ is a *normal subgroup of* $\mathcal{G}$ if $\theta \in \mathcal{G} \Rightarrow \theta^{-1}\theta_e\theta \in \mathcal{G}_e^* \; \forall \; \theta_e \in \mathcal{G}_e^*$). Unfortunately, $\mathcal{G}_e^*$ is not a normal subgroup. For, let $\theta \in \mathcal{G}$ and $\theta_e \in \mathcal{G}_e^*$ be defined by

$$\begin{aligned}
\theta(x) &= \begin{cases} 1-x & if\ 0 \leq x < 1/2 \\ x - 1/2 & if\ 1/2 \leq x \leq 1 \end{cases} \\
\theta_e(x) &= \begin{cases} .01x & if\ 0 \leq x < .8 \\ .008 + 4.96(x - .8) & if\ .8 \leq x \leq 1 \end{cases}
\end{aligned}$$

Let $t = .4$ and $s = .3$. Thus $\theta_e(s) < \theta_e(t)$, but $(\theta^{-1}\theta_e\theta)(s) = .507 > .506 = (\theta^{-1}\theta_e\theta)(t)$, thus $\theta^{-1}\theta_e\theta \notin \mathcal{G}_e$.

Thus the approach we follow in the continuum case is that we start with a group of automorphisms $\tilde{\Theta}$ each of whose elements generates a distinct order. Then we define, $\mathcal{G}_e$ as follows:

$$\mathcal{G}_e = \left\{\theta \in \mathcal{G} \mid \tilde{\theta}^{-1}\theta\tilde{\theta} \sim e \forall \tilde{\theta} \in \tilde{\Theta}\right\}$$

and let us define Θ, the group of Borel automorphisms with respect which our random order operator is to be symmetric by

$$\Theta = \left\{\theta \in \mathcal{G} \mid \theta^{-1}\theta_e\theta \in \mathcal{G}_e \forall \theta_e \in \mathcal{G}_e\right\} \tag{31}$$

Note that Θ includes $\mathcal{G}_e$ and $\tilde{\Theta}$ and that Θ is a group, and $\mathcal{G}_e$ is a normal subgroup of Θ, and $\tilde{\Theta}$ can be identified with the factor group $\Theta/\mathcal{G}_e$, each element of $\tilde{\theta} \in \tilde{\Theta}$ belongs to a distinct right coset which we will also denote by $\tilde{\theta}$.

Now on we will identify $\Theta/\mathcal{G}_e$ with $\tilde{\Theta}$ and define the marginal contribution function, $\phi^{\tilde{\theta}}V$ for an order represented by the right coset $\tilde{\theta} \in \tilde{\Theta}$ in a game $V \in G_I$ by $\phi^{\tilde{\theta}}V \equiv \phi^{\theta}V$, where θ is any member of the right coset $\tilde{\theta}$. The following proposition shows that $\phi^{\tilde{\theta}}V$ is well defined.

Proposition 5. $\theta', \theta \in \tilde{\theta}$ if and only if $\succ_{\theta'} \equiv \succ_{\theta}$. Moreover, for a game V if $\phi^{\theta'}V$, and $\phi^{\theta}V$ both exists then they are equal.

Let $\left(\tilde{\Theta}, \mathcal{A}_{\tilde{\Theta}}, \Gamma\right)$ be a measurable group of random orders. Denote by

$$(\Phi_\Gamma V)(S) = \int_{\tilde{\Theta}} (\phi^{\tilde{\theta}}V)(S) d\Gamma(\tilde{\theta}), \; \forall \; V \in G_I \; S \in \mathcal{B}_I \tag{32}$$

To be able to define value of a game V using random orders similar to finite case, we should have that for all random orders generated by members of $\theta \in \tilde{\Theta}$, $\phi^\theta V$ exists on $\mathcal{B}_I$. There are examples of games in BV for which $\phi^\theta V$ does not exist for some $\theta \in \Theta$, (see , p.95 for an example). Aumann and Shapley showed, however, that for all games in PNA,

Thus we restrict our analysis to the following linear subspace of G_I:

$$LOR\tilde{\Theta} = \{V \in G_I \mid (32) \text{ exists and well defined for all } S \in \mathcal{B}_I\}$$

Proposition 6. LOR$\tilde{\Theta}$ is a linear symmetric subspace of G_I.

PROOF: It is easy to check that LOR$\tilde{\Theta}$ is a linear space. We show that it is symmetric. Let $\pi \in \Theta$, and $V \in$ LOR$\tilde{\Theta}$. We want to show that $\pi^* V \in$ LOR$\tilde{\Theta}$. Let $\theta \in \Theta$. We claim that $\phi^\theta(\pi^* V)$ exists and is given by $\phi^{\theta\pi} V$. To prove this, note that for all $t \in I$,

$$\begin{aligned}
(\phi^\theta \pi^* V)(\ I(s, \succ_\theta)\ ) &= (\pi^* V)(\ I(s, \succ_\theta)\ ) \\
&= V\pi^{-1}\ (\ I(s, \succ_\theta)\ ) \text{ by definition of } \pi^* \\
&= V\left(I\left(\pi^{-1}(s), \succ_{\theta\pi}\right)\right) \text{ by lemma 1} \\
&= (\phi^{\theta\pi} V)\left(I(\pi^{-1}(s), \succ_{\theta\pi})\right) \\
&= ((\phi^{\theta\pi} V)\pi^{-1})\,(I(s, \succ_\theta) \text{ by lemma 1}
\end{aligned}$$

Since they agree on the initial segments $\mathcal{I}_\theta$, they agree on $\mathcal{B}_I$. Thus the measure $\phi^\theta \pi^* V$ exists whenever the measure $\phi^{\theta\pi} V \pi^{-1}$ exists. Since $\theta\pi \in \Theta$ and $V \in$ LOR$\tilde{\Theta}$, $\phi^{\theta\pi} V$ exists, and since $\phi^{\theta\pi} V \pi^{-1}$ is also a measure whenever $\phi^{\theta\pi} V$ is a measure, we have shown that the measure $\phi^\theta \pi^* V$ exists.

Q.E.D.

Linearity, positivity, symmetry, and efficiency of Φ_Γ on $LOR\tilde{\Theta}$ are defined exactly in the same way as in the case of finite players. As in finite case, when Φ_Γ is linear, positive and symmetric on a symmetric subspace $Q \subset LOR\tilde{\Theta}$, Φ_μ is said to be a *random order semi-value operator* on Q. An efficient random order semi-value operator on Q is said to be *a random order value* operator on Q. Note that our definition of value operator is the same as in Aumann and Shapley whereas our definition of semi-value operator differs from Dubey et al., 1981 in that our operator is not required to be a projection.

Theorem 3. Let $\left(\tilde{\Theta}, \mathcal{A}_{\tilde{\Theta}}, \Gamma\right)$ be a measurable group of random orders, then Φ_Γ defined in (32) is a random order semi-value operator on $LOR\tilde{\Theta}$ with respect to the symmetry of the group of Borel automorphisms Θ defines in (31). If furthermore, Γ is a probability measure, then Φ_Γ is a random order value operator on $LOR\tilde{\Theta}$.

PROOF: It is easy to show that Φ_Γ is linear and positive. We show that right invariance of Γ implies that Φ_Γ is symmetric. Note that

$$\begin{aligned}
\Phi_\Gamma(\pi^* V) \qquad (S) &= \int_\Theta \phi^{\tilde{\theta}}(\pi^* V)(S) d\Gamma(\tilde{\theta}) \\
&= \int_\Theta (\phi^{\tilde{\theta}\pi} V)\left(\pi^{-1}(S)\right) \, d\Gamma(\tilde{\theta}), \text{ by } (30) \\
&= \int_\Theta (\phi^{\tilde{\theta}\tilde{\pi}} V)\left(\pi^{-1}(S)\right) \, d\Gamma(\tilde{\theta}\tilde{\pi}) \text{ since } \mu \text{ is right invariant} \\
&= (\Phi_\Gamma V)(\pi^{-1}(S)) \\
&= \pi^*(\Phi_\Gamma V)(S), \; S \in \mathcal{B}_I, \text{ and } V \in \mathrm{LOR}\tilde{\Theta}
\end{aligned}$$

where $\pi \sim \tilde{\pi} \in \tilde{\Theta}$. Hence, $\Phi_\Gamma \pi^* = \pi^* \Phi_\Gamma$. It is easy to note that if Γ is also a probability measure then Φ_Γ is efficient and hence a random order value operator.

Q.E.D.

Definition 3. : A measurable group, $(\Theta, \mathcal{A}_\Theta, \Gamma)$ is a *standard measurable group* if $(\Theta, \mathcal{A}_\Theta, \Gamma)$ is (measure theoretically) isomorphic to a borel subset of a complete and separable metric group.[2]

Theorem 4. Let Φ_Γ be a random order semi-value operator on $LOR\tilde{\Theta}$ with respect to the standard measurable group $\left(\tilde{\Theta}, \mathcal{A}_{\tilde{\Theta}}, \Gamma\right)$, then Φ_Γ is unique up to scalar multiplication, i.e., if there exists another random order semi-value operator $\Phi_{\Gamma'}$, then $\Phi_\Gamma = c\Phi_{\Gamma'}$ for some constant $c \in \Re$, and hence a random-order value operator is unique when it exists.

[2] The standardness assumption implies Hausdorff separation axiom of the underlying space and the assumption enriches the space of measurable functions. A weaker separation notion, defined purely measure theoretically is as follows: A measurable group, $(\Theta, \mathcal{A}_\Theta, \Gamma)$ is *separated* if $\forall\theta \in \Theta, \theta \neq e$, there exists $E \in \mathcal{A}_\Theta$ such that $0 < \Gamma(E) < \infty$ and $\Gamma(E\theta \triangle E) > 0$. Although it is possible to carry out most of the analysis with separated measurable group of random orders, to simplify exposition, we make standardness assumption instead.

xxxx

Remark 7. Aumann and Shapley [1974] proved uniqueness of the axiomatic value operator on bv′NA; Dubey et al. proved uniqueness up to scalar multiplication of semi-value operator on pNA by using the topological properties, (namely, the internality) of those spaces. However, in our random order approach uniqueness is obtained from the properties of Haar measure on the measurable group of random orders.

4 Choices of the sub-groups of random orders and the existence of a random order semi-value operator

So far, we have assumed that a standard measurable group of random orders, $\left(\tilde{\Theta}, \mathcal{A}_{\tilde{\Theta}}, \Gamma\right)$, is given. We now provide a general procedure to construct standard measurable groups of random orders.

From the Haar measure theorem (Halmos, 1950, Theorem B, p.254 and Parthasarathy, 1977, proposition 54.2) and Mackey-Weil theorem (, proposition 65.4, Remark 56.5, the existence of a standard measurable group is equivalent to the existence of a locally compact second countable metric topology on $\tilde{\Theta}$ such that it is also a topological group.

Drawing from the theory of unitary representations of groups, we first provide general procedure to equip metric topology on the group of random orders so that $\tilde{\Theta}$ is a second countable metric group. Random order semi-value (respectively value) operator could then be constructed with respect to any locally compact (respectively compact) subgroup of it. We also provide a few examples of compact subgroups of random orders.

4.1 Choice of $\tilde{\Theta}$

Recall that the elements of $\tilde{\Theta}$ are equivalence classes, each representing a measurable random order. The structure of this set is not known in general. Note, however, that since a Lebesgue measure preserving automorphism, $\theta(x)$ is almost everywhere linear with $\theta'(x) = \pm 1$ except in a set of Lebesgue measure zero, each such automorphism generates a distinct random order on I. This motivates us to restrict the set of measurable random orders to the following set.

$$\tilde{\Theta} = \quad \text{set of all Lebesgue measure preserving Borel automorphisms}$$

It can be easily shown that the above $\tilde{\Theta}$ is a subgroup of the group of Borel automorphisms that we have considered earlier.

4.2 Choice of topology on $\tilde{\Theta}$

Our purpose is to give a topology to $\tilde{\Theta}$ such that it is a topological group. There are several ways $\tilde{\Theta}$ could be given a topology, such as topology of point-wise convergence or that of uniform convergence. However, the group operation may not be continuous with respect to these topologies. Following the techniques used in the literature of unitary representations of groups, we achieve this goal by embedding $\tilde{\Theta}$ homomorphically into the group of unitary bounded operators as follows:

Let $L_2(I)$ be the Hilbert space of square integrable functions with respect to Lebesgue measure on $(I, \mathcal{B}_I)$. Let $\mathcal{U}$ denote the set of all operators U on the Hilbert space $L_2(I)$, such that U is onto and U is isometric, i.e. $(U(f), U(g)) = (f, g)$, $f, g \in L_2(I)$ where $(,)$ is the inner-product operation of $L_2(I)$. Such an operator U of $L_2(I)$ is known as *unitary operator*. It is easy to see that with respect to the strong operator topology, i.e., metric of the Banach space of bounded operators on $L_2(I)$, $\mathcal{U}$ is a complete, separable metric space. With respect to the weak operator topology on $\mathcal{U}$, i.e., the weakest topology on $\mathcal{U}$ with respect to which the maps

$$U \to (Uf, g)$$

are continuous for all fixed $f, g \in L_2(I)$; it can be shown that the map $(U, V) \to UV^{-1}$ from $\mathcal{U} \times \mathcal{U}$ onto $\mathcal{U}$ is continuous, i.e., with respect to weak operator topology, $\mathcal{U}$ is a topological group, known as the *unitary group* of the Hilbert space $L_2(I)$. But since on $\mathcal{U}$, the strong operator topology and weak operator topology coincide (Halmos [1956,p.62]), $\mathcal{U}$ is a complete, separable metric group.

For $\theta \in \tilde{\Theta}$, we define an operator $U(\theta) : L_2(I) \to L_2(I)$ by

$$(U(\theta)f)(x) = f(\theta(x)), \; f \in L_2(I)$$

It can be shown that $U(\theta)f \in L_2(I)$, and $U(\theta)$ is a unitary operator on $L_2(I)$ and that the map $U : \tilde{\Theta} \to \mathcal{U}$ is a homomorphism, i.e., for all θ_1,$\theta_2 \in \tilde{\Theta}$ $U(\theta_1 \circ \theta_2) = U(\theta_1)U(\theta_2)$, with the identity of $\tilde{\Theta}$ mapped into identity of $\mathcal{U}$. The above imbedding of the group $\tilde{\Theta}$ in the unitary group $\mathcal{U}$ is known as the *unitary representation of the group* $\tilde{\Theta}$. Thus with respect to the relative topology in $\mathcal{U}$, $\tilde{\Theta}$ is a second countable metric group. We may now choose a group of measurable random orders to be any subgroup such that it is a locally compact space with respect this topology. We do not know if $\tilde{\Theta}$ itself is locally compact or not.

Example 3. Let Θ_0 be the group of all continuous automorphisms of I, i.e., the group of homeomorphisms of I. There are only two orders on I that are compatible with these automorphisms, namely the standard order, represented by $\theta_0^0(x) = x$ for all $x \in I$ and the reverse order represented by $\theta_0^1(x) = 1 - x$, for all $x \in I$. Both of these are measurable random orders in the sense defined earlier. We denote this set of random orders by $\tilde{\Theta}_0$. In this case, $\mathcal{G}_e$ is very large. It is trivial to note that θ_0^1 is the identity and $\theta_1^1\theta_1^1 = \theta_0^1$. Hence it is a group. Since $\tilde{\Theta}_0$ is finite it is compact with respect to any topology. Since $\tilde{\Theta}_0$ has only two elements, the unique invariant (both left and right) assigns probability mass of 1/2 to each element. Thus, the random order value operator on $LOR\tilde{\Theta}_0$ is given by

$$(\Phi_\Gamma V)(S) = \sum_{\tilde{\theta} \in \tilde{\Theta}_0} \frac{1}{2} \cdot \left(\phi^{\tilde{\theta}} V\right)(S) \tag{33}$$

The above Φ_Γ is symmetric with respect to the group of all homeomorphisms of I.

Example 4. For a fixed $n \geq 0$, let us consider a partition of the closed interval $[0, 1]$ into 2^n sub-intervals of equal size as follows:
$\left[0, \frac{1}{2^n}\right), \left[\frac{1}{2^n}, \frac{2}{2^n}\right), \ldots., \left[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}\right), \left[\frac{2^n-1}{2^n}, \frac{2^n}{2^n}\right],$

Let the above sub-intervals be denoted respectively as I_0, I_1, ..., I_{2^n-2}, I_{2^n-1}. Let us consider $I \times I$, as a square in $\Re^2_+$. Then the above sub-intervals on the x-axis and y-axis determine $(2^n.2^n)$ square boxes, each of length $1/2^n$. For $n = 2$, these square boxes are shown in panel (a) of figure (1).

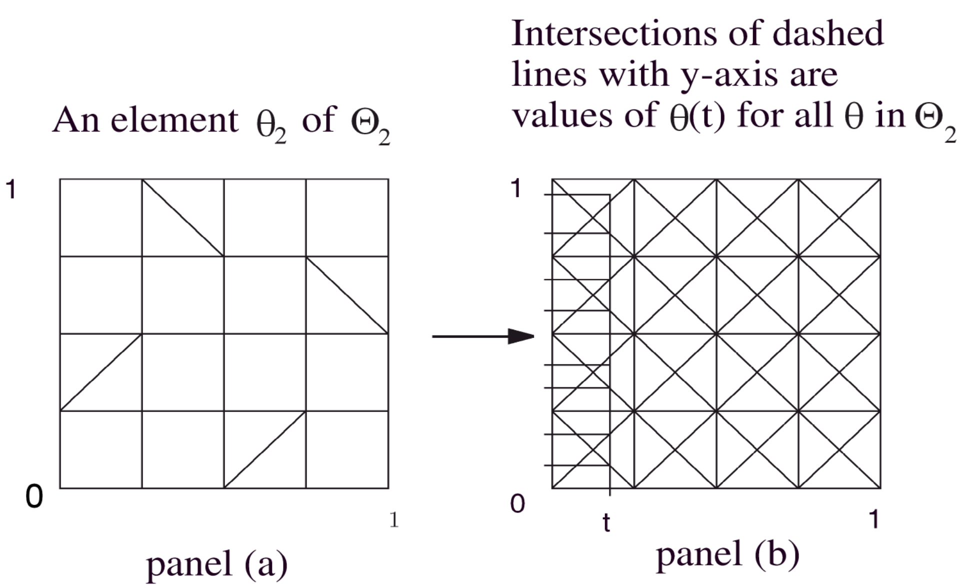

Figure 1:

Let $N_n = \{0, 1, 2, ..., 2^n - 1\}$, and let $\pi_n : N_n \to N_n$ be such that π_n is one-one and onto, i.e., π_n is a permutation of N_n. We use π_n to assign each subinterval I_i, to the subinterval $I_{\pi_n(i)}$ $i = 0, 1, 2, ..., 2^n - 1$. Let $\mathcal{O}_n : N_n \to \{+1, -1\}$. $\mathcal{O}_n(i)$ denotes the orientation of the interval I_i, $i \in N_n$ on the y-axis. For any permutation π_n and for each given orientation $\mathcal{O}_n$ of the sub-intervals, we define a Lebesgue measure preserving automorphism as follows:

$$\theta_n(x) = \begin{cases} \frac{\pi_n(k)}{2^n} - \frac{k}{2^n} + x & if\ x \in I_k\ and\ \mathcal{O}_n(\pi_n(k)) = +1 \\ \frac{\pi_n(k)+1}{2^n} - \frac{k}{2^n} - x & if\ x \in I_k\ and\ \mathcal{O}_n(\pi_n(k)) = -1 \\ & k = 0, 1, ...2^n - 1 \end{cases} \tag{34}$$

In panel (a) of figure 1, we have shown the graph of $\theta_2(x)$ corresponding to the permutation, $\pi_2(1) = 2$, $\pi_2(2) = 4$, $\pi_2(3) = 1$ and $\pi_2(4) = 3$, and the orientation, $\mathcal{O}_2(1) = +1$, $\mathcal{O}_2(2) = -1$, $\mathcal{O}_2(3) = +1$, and $\mathcal{O}_2(4) = -1$.

Let

$$\tilde{\Theta}_n = \left\{ \begin{array}{c} \theta_n : I \to I, \text{ defined by (34)} \mid \pi_n \text{ is a permutation of } N_n \\ \text{and } \mathcal{O}_n \text{ is an orientation of the } 2^n \text{sub-intervals} \end{array} \right\}$$

There are $(2^{2^n}.2^n!)$ total number of elements in $\tilde{\Theta}_n$. It can be shown easily that each θ_n is a Lebesgue measure preserving automorphism and that $\tilde{\Theta}_n$ is a subgroup of $\tilde{\Theta}$.

Let us examine the kind of randomization of the players that are performed by the random orders in $\tilde{\Theta}_n$ in example 4. For illustration purpose, let us consider the automorphism θ_2 that is depicted in panel (a) of figure 1. Note that the set of players before player t, $t \in I$ in the random order $\theta_2 \in \tilde{\Theta}_2$ is given by

$$I\left(t, \succ_{\theta_2}\right) = \begin{cases} [0, t) \cup I_3 & if\ t \in I_1 \\ I_1 \cup (t, \frac{1}{2}) \cup I_3 \cup I_4 & if\ t \in I_2 \\ [\frac{1}{2}, t) & if\ t \in I_3 \\ I_1 \cup I_3 \cup (t, 1] & if\ t \in I_4 \end{cases}$$

Note that the nature of randomization produced by an element of $\tilde{\Theta}_n$ depends on the permutation π and the orientation $\mathcal{O}$. Let us fix a $t \in I$ and suppose $t \in I_1$. Consider the initial segments of player t in each of the random orders $\theta_n \in \tilde{\Theta}_n$, that has the same value for t, say $\theta_n(t) = t_0$. All of these random orders will have either positive orientation or negative orientation. Let us assume that they have positive orientation. Let us denote by [x], $x \in \Re$, as the greatest integer in x. The way the initial segments are randomized by these random orders is that $[\frac{2^n}{t_0}]$ sub-intervals from the set of all sub-intervals except I_1 are randomly selected and then placed before the set of points [0,t) in all possible permutations. For very large n, size of each sub-intervals is very small, and hence for large n, all these $\theta_n' s$ with fixed value of $\theta_n(t) = t_0$ are placing almost any infinitesimally small subintervals of I that can fit in an interval of size $[0, t_0]$. The size of the interval also vary as we vary t_0 in the set $T_n = \{\theta(t) \mid \theta \in \tilde{\Theta}_n\}$.

In panel (b) of figure (1), we have graphed all the elements of $\tilde{\Theta}_2$. The set T_2 is shown as the intersection of the dash lines with the y-axis. It is trivial to note that $\tilde{\Theta}_n \subset \tilde{\Theta}_{n+1}$ and as $n \to \infty$, the number of elements in T_n becomes large and are spread uniformly over I.

Since each $\tilde{\Theta}_n$ is finite, it is a compact group with respect to any topology. Thus in each $\tilde{\Theta}_n$ we can define a random order value operator Φ_{Γ_n}using the unique (right) invariant Haar probability measure, Γ_n. However, since our purpose is to obtain the largest locally compact topological subgroup of $\tilde{\Theta}$, it is clear that $\check{\Theta} \equiv \cup_{n=1}^{\infty}\tilde{\Theta}_n = \lim_{n\to\infty} \tilde{\Theta}_n$ is a countably infinite locally compact subgroup of $\tilde{\Theta}$,and thus admits right invariant Haar measure say Γ. Applying theorem 3, we can construct random order semi-value operator Φ_Γ with respect to the random orders in $\check{\Theta}$.

Although for some game $V \in LOR\check{\Theta}$, it is possible that $(\Phi_{\Gamma_n} V)(S) \to \Phi[V](S)$, as $n \to \infty$ for all $S \in \mathcal{B}_I$, where $\Phi[V](.)$ is a finitely additive set function which satisfies all the axioms of Shapley value, we can not, however, assure that $\Phi[V]$ coincides with a random order value or even semi-value operator $\Phi_\Gamma V$ for some right invariant measure Γ on $\check{\Theta}$. However, the above construction can be used to *conjecture* an *alternative approach to random order value* similar to the mixing value approach of Aumann and Shapley as follows: Let MIX$\tilde{\Theta}$ be the space of games $V \in LOR\tilde{\Theta}$ for which the above sequence, $(\Phi_{\Gamma_n} V)(S)$ converges, to the finitely additive measure $\Phi[V](S)$, then it is easy to verify that MIX$\tilde{\Theta}$ is a linear space and Φ is linear operator on MIX$\tilde{\Theta}$. If Φ also satisfies the other axioms of value. We have not followed this line of research.

We have shown elsewhere (Raut, 1997), however, that when $\tilde{\Theta}_n' s$ are restricted to satisfy certain projection condition, the sequence $\{\Phi_{\Gamma_n} V\}_0^\infty$ converges for certain type of measure valued games in pNA. Furthermore, the limit is a random order value operator with respect to an uncountably large group of Lebesgue measure preserving automorphisms and the random order Shapley value admits a diagonal formula which coincides with the diagonal formula given by Aumann and Shapley, 1974 for such games.

References

Aumann, R. J. and L. S. Shapley (1974). Values of Non-Atomic Games. Princeton University Press (cit. on pp. 2, 3, 5, 6, 20, 23, 29).

Billera, L. J., D. C. Heath, and J. Raanan (1978). Internal Telephone Billing Rates—A Novel Application of Non-Atomic Game Theory, *Operations Research*, **26**, no. 6, 956–965. DOI: 10.1287/opre.26.6.956 (cit. on p. 2).

Dubey, P., A. Neyman, and R. J. Weber (1981). Value Theory Without Efficiency, *Mathematics of Operations Research*, **6**, no. 1, 122–128. DOI: 10.1287/moor.6.1.122 (cit. on pp. 2, 3, 23).

Halmos, P. R. (1950). Measure theory. Vol. 2. van Nostrand New York. DOI: 10.1007/978-1-4684-9440-2 (cit. on pp. 3, 17, 25).

Parthasarathy, K. R. (1977). Introduction to probability and measure, (cit. on pp. 3, 17, 25).

Raut, L. K. (1981). Random Order Approach to Semi-values of Non-atomic games, *Indian Statistical Institute Discussion Paper*, no. 8108 (cit. on p. 3).

Raut, L. K. (Mar. 1997). Construction of a Haar measure on the projective limit group and random order values of non-atomic games, *Journal of Mathematical Economics*, **27**, no. 2, 229–250. DOI: 10.1016/s0304-4068(96)00753-7 (cit. on pp. 4, 29).

Roth, A. (1988). The Shapley Value: Essays in Honor of Lloyd S. Shapley. Cambridge University Press (cit. on p. 2).

Samet, D. and Y. Tauman (July 1982). The Determination of Marginal Cost Prices under a Set of Axioms, *Econometrica*, **50**, no. 4, 895. DOI: 10.2307/1912768 (cit. on p. 2).

Shapley, L. S. (1953). *A value for n-person games*. Tech. rep. 28, 307–317 (cit. on pp. 2, 3, 5, 6).

Shapley, L. S. and L. K. Raut (1981). Theory of Games and its Applications to Economics and Politics. 10. Macmillan India (cit. on p. 5).