RANDOM ORDER APPROACH TO SEMI-VALUES OF NON-ATOMIC GAMES

By
L.K. Raut

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L.K. Raut
Indian Statistical Institute
7, SJS Sansanwal Marg
New Delhi-110016, India

1. Introduction

Shapley (1953) originally gave two equivalent approaches – one is axiomatic and the other one is random order – to the valuation of games with finite number of players. In social sciences, games with infinitely many players – each with negligible effect on the total outcome – occur very frequently. Asymptotically, these can be represented by games with continuum of players, each player having negligible weight in changing the outcome, in the same sense of approximation as used in physics to approximate infinitely many liquid drops in a container with continuum of points.

Analogue of the axiomatic approach of finite games to non-atomic games is a well established area of game theory, whereas for random order approach, its adaptation is only upto proving the impossibility (See section 2) for a very small class of non-atomic games, provided we take the underlying player space to be isomorphic to $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the borel σ-algebra of $[0,1]$ with the understanding that the sets in $\mathcal{B}$ are the possible coalitions. We shall have the possibility of random order type of approach to some semi-values on a very big space of games. Semi-values have economic importance. Jamet and Tauman (1979) used it to characterise a class of price mechanism.

2. Random Order Approach : Finite and Continuum Cases

Let us first consider the finite case. Let $N = \{1,2,\ldots,n\}$ be a fixed set of n players and let $\mathcal{B}$ be the power set of N.

Let

$$G_N = \{v : \mathcal{B} \to \mathbb{R} \text{ s.t. } v(\phi) = 0\}$$

It is easy to varify that G_N is a linear space over $\mathbb{R}$ and let

$$FA = \{v \in G_N \mid v \text{ is additive}\}$$

FA is a linear subspace of G_N. For any subspace $Q \subset G_N$, denote by $Q^+ = \{v \in Q \mid S,T \in \mathcal{B}, S \supset T \Rightarrow v(s) \geq v(T)\}$

The games in Q^+ will bo called the monotonic games in Q. Let

$$\Theta = \{\theta : N \to N \mid \theta \text{ is 1-1, onto}\}$$

For each θ in Θ, define the linear operator θ_* on G_N by

$$(\theta_* v)(S) = v(\theta^{-1}S)$$

One can verify easily that $\theta_* G_N \subset G_N$. We shall call a subspace $Q \subset G_N$ to be symmetric if for all $\theta \in \Theta$, $\theta_* Q \subset Q$.

Definition 1 An operator $\Phi : G_N \to FA$ will be called a value on G_N if it satisfies the following axioms:

Axiom 1 (Linearity)

$$\phi(\alpha v_1 + v_2) = \alpha\phi(v_1) + \phi(v_2) \text{ whenever } v_1, v_2 \in G_N,\ \alpha \in \mathbb{R}$$

Axiom 2 (symmtry)

$$\phi\theta_* = \theta_*\phi \qquad \forall\, \theta \in \Theta$$

Axiom 3 (Positivity)

$$\phi G_N^{+} \subset FA^{+}$$

Axiom 4 (Efficiency)

$$(\phi v)(N) = v(N) \quad \forall \; v \in G_N$$

For intuitive meanings to above axioms see Shapley (1979). It is proved that there exists a unique value on G_N. We shall call an operator linear, positive, efficient if it satisfies respectively axioms 1,3,4 on a linear subspace Q of G_N; if Q is a symmetric subspace and an operator satisfies axiom 2, we shall call it a symmetric operator.

Definition 2 A semi-value on a symmetric subspace of G_N is a linear, symmetric, positive operator from that subspace to FA.

A random order on N is a transitive, irreflexive and complete binary relation $>_R \subset N \times N$. Let Ω be the set of all such random orders. It is easy to see that all random orders on N can be induced by the permutations and vice versa in the following way:

$$>_R \; ; \; i >_R j \iff \theta(i) > \theta(j) \; , \quad \theta \in \Theta \; \ldots \qquad (E.1)$$

An initial segment in the random order $>_R$ is a set of the form:

$$I(s, >_R) = \{ j \in N : s >_R j \} \quad s \in N$$

This is the set of players who are before s in the ordering $>_R$. A marginal contribution function (set function) in a random order $>_R$ for a game v is a measure $\varphi^R v$ on $(N, \mathcal{B})$ which satisfies

$$(\varphi^R v)(i) = v(I(i,>_R) \cup \{i\}) - v(I(i,>_R)).$$

We note that there is a 1-1 and onto correspondence between Ω and Ⓗ satisfying E.1 and hence we can identify each $>_R$ with its associated permutation θ in Ⓗ. In the random order, induced by θ, we shall identity $\theta(i)$ by player i. With this convention, we have

$$I(s,>_R) = \{ j \in N: \theta(j) < \theta(s) \} = I(s,\theta) \text{ say.}$$

Let $\mathcal{B}_\Omega$ be the power set of Ω and let ω be a probability measure on $(\Omega, \mathcal{B}_\Omega)$. Define the operator $\varphi_R : G_N \to FA$ by

$$(\varphi_R v)(S) = \int (\varphi^\theta v)(S) d\omega(\theta) \tag{E.2}$$

<u>Proposition 3</u> ϕ_R is a value on G_N if ω is a right invariant probability measure.

<u>Proof</u> : Linearity, positivity and efficiency follow easily.
Symmetry:
Let $\pi \in$ Ⓗ and $v \in G_N$ be arbitrarily fixed. We want to show that $\phi_R \pi^* = \pi^* \varphi_R$. Now note that

$$\begin{aligned}(\varphi^\theta(\pi^* v))(i) &= (\pi^* v)(I(i,\theta) \cup \{i\}) - (\pi^* v)(I(i,\theta)) \\ &= v(I(\pi^{-1}(i), \theta\pi) \cup \pi^{-1}\{i\}) - v(I(\pi^{-1}(i),\theta\pi)) \\ &= (\varphi^{\theta\pi} v)(\pi^{-1}(i)) \quad \forall \; i \in N\end{aligned}$$

Hence

$$(\varphi^\theta(\pi^* v))(S) = (\varphi^{\theta\pi} v)(\pi^{-1} S) \quad \forall \; S \in \mathcal{B}$$

Now,

$$\begin{aligned}\varphi_R(\pi^* v)(S) &= \int (\varphi^\theta(\pi^* v))(S)\, dw(\theta) \\ &= \int (\varphi^{\theta\pi} v)(\pi^{-1} S) dw(\theta\pi) \quad \text{Since } \omega \text{ is right invariant} \\ &= (\varphi_R v)(\pi^{-1} S) \\ &= \pi^*(\varphi_R v)(S) \quad \forall \; S \in \mathcal{B} \text{ and } \forall \; v \in G_N\end{aligned}$$

Thus $\varphi\pi^* = \pi^*\varphi$ Q E D

Now let us turn to the non-atomic case

Notations and Definitions:

$I = [0,1]$: player set

$\mathcal{B}$ = borel σ-algebra of I : set of possible coalitions

A game is a set function $v: \mathcal{B} \rightarrow \mathbb{R}$ with $v(\phi) = 0$. A game is said to be monotonic if $S, T \in \mathcal{B}$, $S \supset T \Rightarrow v(S) \geq v(T)$. A game is said to be of bounded variation if $v = u - w$, where u and w are both monotonic games. Denote by

BV = set of all games of bounded variations.

It is easy to varify that BV is a linear space and with the following norm it is a Banach space.

$$||v|| = \inf \{ u(I) + w(I) : v = u - w,\ u \text{ and } w \text{ are monotonic games} \}$$

Let FA = set of finitely additive set functions from BV

NA = set of non-atomic measures on $(I, \mathcal{B})$

pNA = $||.||$ - closure of the algebra generated by the powers of non-atomic measures

$\mathcal{H}$ = set of all borel automorphisms of $(I, \mathcal{B})$.

A random order on I is a transitive, irreflexive and complete order $>_R$ which also satisfies the following: The family

$$\mathcal{I} = \{ I(s, >_R) \mid s \in \{-\infty\} \cup I \cup \{\infty\} \ (= \bar{I}, \text{ say}) \}$$

With the convention of $I(-\infty, >_R) = \phi$, $I(\infty, >_R) = I$, generates the borel σ-algebra of I. Let

$\mathcal{R}$ = set of all random orders.

As in the finite case, we define the marginal contribution function in a random order $>_R$ for a game v to be a measure $\phi^R v$, on (I,) such that $(\varphi^R v)(I(s,>_R)) = v(I(s,>_R))$. Note that for a game v and a random order $>_R$ if $\varphi^R v$ exists it is unique. It is shown for all v in pNA and $>_R$ in φ^R_v exist.

If random order approach of finite case could have been adopted for even a smallest, economically important class of games, pNA, then we could get a σ-algebra of and a probability measure on it such that E.2 defines a value, but we have the following.

Theorem 4 (Aumann and Shapley)

There is no σ-algebra of with a probability measure on it such that the positive, efficient linear operator φ:pNA $\longrightarrow$ FA defined by

$$(\varphi v)(S) = \int (\varphi^R v)(S)\, dw(R)$$

is symmetric.

Proof. See Aumann and Shapley (1974, Theorem D)

3. Semi Value

We shall give here a random order type of approach to semi-value on a quite large space, ORΘ, of games. Let $\theta \in \Theta$. identify $\theta(x)$ as the player x in the ordering induced by θ. Define

$$I(s,\theta) = \{x \in I: \theta(x) < \theta(s)\}$$

Let $\varphi^\theta v$ be a measure on $(I, \mathcal{B})$, satisfying

$$(\varphi^\theta v)(I(s,\theta)) = v(I(s,\theta)) \quad \forall s \in \bar{I}$$

Note that whenever $\varphi^\theta v$ exists it is unique. Let

$$OR\,\Theta = \{ v \in Bv : \varphi^\theta v \text{ exists } \forall \ \theta \in \Theta$$

Proposition 5

$OR\,\Theta$ is a linear symmetric subspace of BV.

Proof Let $v_1, v_2 \in OR\,\Theta$ and $\theta \in \Theta$.

$$(\varphi^\theta v_1 + \varphi^\theta v_2)(I(s,\theta)) = (\varphi^\theta v_1)(I(s,\theta)) + (\varphi^\theta v_2)(I(s,\theta))$$

$$= v_1(I(s,\theta)) + v_2(I(s,\theta))$$

$$= (v_1 + v_2)(I(s,\theta)).$$

Hence $(\varphi^\theta v_1 + \varphi^\theta v_2)$ equals to infact $\varphi^\theta(v_1+v_2)$ and thus $v_1+v_2 \in OR\,\Theta$ Similarly $\alpha \in \Theta$ and $v \in OR\,\Theta$ $\alpha v \in OR\,\Theta$.

Let $\pi \in \Theta$ be arbitrarily fixed. To show $\pi^* \, OR\,\Theta \subset OR\,\Theta$. Let $\theta \in \Theta$. Note that $(\varphi^\theta \pi^* v)(S) = (\varphi^{\theta^\pi} v)\pi^{-1}(S)$ and since $(\varphi^{\theta^\pi} v)\pi^{-1}$ is a measure, so, $\varphi^\theta \pi^* v$ exists and hence $\pi^* v \in OR\,\Theta$

Q E D

Theorem 6

There exists a σ-algebra, $\mathcal{B}_\Theta$ of Θ and a measure on $(\Theta, \mathcal{B}_\Theta)$ such that the operator $\varphi : OR\,\Theta \to FA$ defined by

$$(\varphi v)(S) = \int_\Theta (\varphi^\theta v)(S) \, dw(\theta)$$

is a semi-value on $OR\,\Theta$.

<u>Proof</u> : Let us adorn Θ with discrete topology. So it is trivial to note that Θ with this topology is a locally compact topological group. Let $\mathcal{B}_{\Theta}$ be the borel σ-algebra of Θ. By Haar measure theorem [Halmos 1964, Theorem B] there exists a regular borel measure which is right-invariant. Let ω be this right invariant borel measure on $(\Theta, \mathcal{B}_{\Theta})$.

Linearity follows from the fact that $(\varphi^{\theta}v)(S)$ is linear in v for all $s \in \mathcal{B}$, $\theta \in \Theta$. To show positivity, let $\theta \in \Theta$, and let v be a monotonic game in OR Θ, then for $s > t$ in the induced order of θ.

$$(\varphi^{\theta}v)(I(s,\theta)\setminus I(t,\theta)) = (\varphi^{\theta}v)(I(s,\theta)) - \varphi^{\theta}v(I(y,\theta))$$

$$= (v(I(s,\theta)) - v(I(t,\theta))$$

$$\geq 0 \quad \text{sin} \quad I(s,\theta) \quad I(t,\theta).$$

Now, the sets of the form $I(s,\theta)\setminus I(t,\theta)$ generate $\mathcal{B}$ on I
Hence, $(\varphi^{\theta}v)(S) \geq 0 \ \forall\ S \in \mathcal{B} \Rightarrow (\varphi^{\theta}v) \in FA^{+} \Rightarrow \varphi v \in FA^{+}$.
To show symmetry, let $\pi \in \Theta$ and $v \in$ OR Θ. Then

$$\begin{aligned}
(\varphi\pi^{*}v(S) &= \int_{\Theta} (\varphi^{\theta}\pi^{*}v)(S)\, d\omega(\theta) \\
&= \int_{\Theta} (\varphi^{\theta\pi}v)(\pi^{-1} S)\, d\omega(\theta\pi) \quad \text{Since is right invariant} \\
&= (\varphi v)(\pi^{-1}S) \\
&= (\pi^{*}\varphi v)(S) \qquad \forall\ S,\ v.
\end{aligned}$$

Hence $\varphi\pi^{*} = \pi^{*}\varphi$

Q E D

<u>Remark</u> By mimicking exactly the same way as in proposition 12.8 of Aumann and Shapley (1974) it can be shown that OR $\Theta \supset$ pNA. It is important to know whether ORΘ contains DIFF (cf Marten 1978) because DIFF is the upto date largest space on which a value exists.

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