ISI LECTURE NOTES

VALUES OF NON-ATOMIC GAMES

By

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LECTURE 1

MATHEMATICAL PRELIMINARIES

We shall develop a body of concepts and results which will be needed, somewhere or other, in our main stream. One can use the word "game" in place of 'set function' interchangably.

Let

$I = [0,1]$

$\mathcal{B}$ = borel σ-algebra of I

Whenever we use any set, it is understood to be a set from $\mathcal{B}$ unless otherwise stated.

Definition 1.1: A set function is a map $v : \mathcal{B} \to \mathbb{R}$ s.t. $v(\phi) = 0$.
v is monotonic if $\forall$ S,T, $S \subset T \implies v(S) \leq v(T)$.

Definition 1.2: v is said to be of bounded variation if there exist monotonic set functions u and w s.t. $v=u-w$. Let

BV = set of all set functions of bounded fariation.

Define a map $||.|| : BV \to \mathbb{R}$ by

$$||v|| = \inf \{u(I)+w(I) : v=u-w, \text{ u and w are monotonic set function}\}$$

One, at once, varifies that $||.||$ is a norm on BV. Call this as BV - norm.

Definition 1.3: A chain Ω is a finite collection of sets. $\{S_i\}_{i=1}^{n} \subset \mathcal{B}$ s.t.
$\phi = S_o \subset S_1 \subset \ldots . \subset S_n = I$.

For a chain Ω and a set function v, define

$$||v||_\Omega = \sum_{i=1}^{n} |v(S_i) - v(S_{i-1})|$$

Note that for each Ω, $||.||_\Omega$ is a "semi-norm". The following theorem establishes the relationship between BV-norm and $||.||_\Omega$:

Proposition 1.4: $v \in BV \iff ||v||_\Omega$ is bounded $\forall \Omega$. Moreover
$||v|| = \sup_\Omega ||v||_\Omega$.

A non-null coalition A is said to be an atom of v if $\forall$ T $\subset$ A either T or A-T is null. A non-atomic set function (game) is a set function without any atom.

Note that if v is a measure then both the definitions are same. Moreover, if μ is measure on $(I, \mathcal{B})$ then μ is non-atomic if and only if $\mu(\{x\}) = 0 \ \forall x \in I$.

Example 1.13: Let μ = Lebesgue measure on $(I, \mathcal{B})$. Then μ is non-atomic. Let, for some positive integer n,

$$v(S) = (\mu(S))^n, \quad \forall S \in \mathcal{B}.$$

It is very easy to varify that v is a non-atomic set function. The following theorem will be used widely in the sequel.

Proposition 1.14 (Lyapunov's theorem): Let $\mu \in (NA)^m$ then $R(\mu) = \{\mu(S) \in \mathbb{R}^m : S \in \mathcal{B}\}$ is compact and convex.

Proof: See (3).

Corollary 1.15: Let $\mu \in (NA)^m$ and let S^1 and $S^o \in \mathcal{B}$ are s.t. $S^1 \supset S^o$. Then there exist a family of sets

$$\{S^\alpha \in \mathcal{B} : 0 \leq \alpha \leq 1\} \text{ s.t.}$$

(i) $\mu(S^\alpha) = \alpha\,\mu(S^1) + (1-\alpha)\,\mu(S^o)$, and

(ii) $\alpha > \beta \implies S^\alpha \supset S^\beta$.

Proof: See A-S(1) lemma 5.4, pp.36.

Proposition 1.16: If $\mu \in NA^1$, then there is an automorphism θ of $(I, \mathcal{B})$ s.t.

$$\theta_* \mu(S) \equiv \mu(\theta S) = \lambda(S) \quad \forall S \in \mathcal{B},$$

where λ is a Lebesgue measure.

Proof: See K.R.Parthasarathy (1977) (2) proposition 26.4 for the proof of a more general result.

Definition 1.17: A linear subspace Q of BV is said to be internal if

$$\|v\|_Q = \inf\{ \text{[illegible]} (T) : u, w \in Q^+ \text{ (not } BV^+\text{) and } v = u - w\}$$
$$= \|v\|_{BV} \quad \forall\, v \in Q.$$

Exercise 1.18: The BV-closure of an internal space is internal. (See A-S pp. 31).

Proposition 1.19: PNA is internal.

Proof The proof will be given in several steps. Let us start with:

Definition 1.20: Let $f: I^n \to \mathbb{R}$. Define, for each $\underline{x} \in I^n$,

$$T_f(\underline{x}) = \sup \sum_{i=1}^{k} |f(\underline{x}_i) - f(\underline{x}_{i-1})| ,$$

where sup is taken over all finite "chains" of the form

$$0 \leq \underline{x}^0 \leq \underline{x}' \leq \ldots \leq \underline{x}^k = \underline{x}.$$

$T_f(\underline{x})$ is called the total variation of f. Note that

$$\underline{x} \leq \underline{y} \implies 0 \leq T_f(\underline{x}) \leq T_f(\underline{y}) \leq \infty .$$

Let $\underline{1} = (1,1,1,\ldots,1)$. If $T_f(\underline{1}) < \infty$, we say that f is of bounded variation.

Lemma 1.21: Let $f \in C^1(I^n)$. Then f is of bounded variation and T_f is continuous on I^n.

Proof: See A-S(1) pp. 48-49. Also see Apostol (4)pp. 132,theorem, 6.14 for function of one variable case.

Now, for each $k > 0$ and m with $1 \leq m \leq 2^k$, define a measure

$$\lambda_m^k (S) = 2^k \lambda(S \cap [\tfrac{m-1}{2^k}, \tfrac{m}{2^k}]),$$

where λ is the Lebesgue measure. Denote by

$$\lambda^k = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_{2^k}^k)$$

Note that $R(\lambda^k) = [0,1]^{2^k}$ and $\forall\, i \neq j \;\; \lambda_i^k \perp \lambda_j^k$, $\forall k$.

Denote by

$$A = \{f \circ \lambda^k : k > 0 \text{ and } f \in C^1(R(\lambda^k)) \text{ with } f(0) = 0\} .$$

Lemma 1.22: A is a sub-algebra of BV.

Proof: Define a linear operator $\psi_k: R(\lambda^k) \to R(\lambda^{k-1})$ by

$$\psi_k(\lambda_1^k, \lambda_2^k, \ldots \lambda_{2^k}^k) = \left(\frac{\lambda_1^k + \lambda_2^k}{2}, \ldots, \frac{\lambda_{2^k-1}^k + \lambda_{2^k}^k}{2} \right)$$

Note that ψ_k is an onto mapping. Define, for $k > m$

$$\psi_{km} = \psi_{m+1} \circ \ldots \circ \psi_{k-1} \circ \psi_k .$$

note that ψ_{km} is a linear map from $R(\lambda^k)$ onto $R(\lambda^m)$ with

$$\psi_{km}(\lambda^k) = \lambda^m .$$

Now, let $u,v \in A$. Then u and v are of the form:

$$u = f \circ \lambda^k, \qquad f \in C^1(R(\lambda^k)) \text{ and } k > 0$$

$$v = g \circ \lambda^m, \qquad g \in C^1(R(\lambda^m)) \text{ and } m > 0$$

w.l.g. assume $k > m$. We want to show $u+v \in A$. That is, we want to produce an $r > 0$ and $h \in R^1(\lambda^r)$ s.t.

$$u+v = h \circ \lambda^r$$

For this, take $h = f + g \circ \psi_{km}$ and $r = k$. Similarly, to show $uv \in A$, take $h = f. g \circ \psi_{km}$ and $r = k$. Other conditions are also easily varified.

Q.E.D

Lemma 1.23: A is internal

Proof: Let $v \in A$. So $v = f \circ \lambda^k$ for some $k > 0$ and $f \in C^1(R(\lambda^k))$ with $f(0) = 0$. We want to show

$$||v||_{BV} = ||v||_A$$

Set $n = 2^k$, $R = R(\lambda^k) = I^n$; and define, for each $x \in R$,

$$f^+(x) = \sup \sum_{i=1}^{k} \max(0, f(x_i) - f(x_{i-1}))$$

$$f^-(x) = \sup \sum_{i=1}^{k} \max(0, f(x_{i-1}) - f(x_i))$$

where sup is taken over all chains of the same type as given in the definition 1.20. Note that

$$T_f = f^+ + f^-$$

$$f = f^+ - f^-$$

Moreover, the above decomposition has the following properties:

1° f^+ and f^- are continuous and T_f is bounded, by lemma 1.21.

2° $||v||_{BV} = T_f(\underline{1}) = f^+(\underline{1}) + f^-(\underline{1})$, since all chains in the definition of T_f can be realised by taking $x^j = \lambda^k(S_j)$ from chains in the definition of $|| \; ||_{BV}$ and vice versa, owing to the mutual singularities of the component measures of λ^k (How?)

Had f^+ and $f^- \in C^1(R)$ then the proof of the lemma could be as simple as upto above. But $f^+, f^- \notin C^1(R)$. The approach to be adopted here is to get $k, h \in C^1(R)$ s.t.

(i) $f = h - k$

(ii) $h(0) = k(0) = 0$, and

(iii) $h(1)$ and $k(1)$ approximate $f^+(\underline{1})$ and $f^-(\underline{1})$.

To that end, denote by $f_i = \frac{\partial f(x)}{\partial x_i}$ and $D = \max_{i,x} | f_i(x) |$. Fix $\varepsilon > 0$ arbitrary and let $\delta > 0$ be s.t.

$$||x-y|| < \delta \implies \max_i |f_i(x) - f_i(y)| < \varepsilon \;\forall\; x,y \in R \text{ and}$$

$\delta < \varepsilon/D$ (it is possible).

Let us define a linear operator from $C(R)$ to $C(R)$ by

$$g(x) \to g^\delta(x) = \int_{y \in R} g((1-\delta)x + \delta y)\, dy$$

$$= \frac{1}{\delta^n} \int \dots \int_{z_i=(1-\delta)x_i}^{(1-\delta)x_i+\delta} \int g(z)\, dz_1, \dots, dz_n$$

The integral above is either Reiman or Lebesgue sense since both are same for continuous functions. Now, note the following:

3°: $g^\delta \in C^1(R)$ even though g is just continuous, which is by a property of Reiman integral.

4° If $g \in C^1(R)$ then $(g^\delta)_i = (1-\delta)(g_i)^\delta$.

$\underline{5^o}$ g is non-decreasing $\Longrightarrow g^\delta$ is also non-decreasing and

$g^\delta(0) \geq g(0)$

$g^\delta(\underline{1}) \leq g(\underline{1})$

$\underline{6^o}$ For every $x \in R$, there exists $y \in R$ s.t. $||x-y|| < \delta$ and $g(y) = g^\delta(x)$. This is true because we are averaging a continuous function around a cube consisting of sides less than δ from x.

$\underline{7^o}$ $\forall\ x \in R$ and $\forall i$, we have $|f_i^\delta(x) - f_i(x)| < \varepsilon$; by applying 6^o on $g = f_i$.

$\underline{8^o}$

$$\left|\frac{\partial(f -f)(x)}{\partial x_i}\right| \leq |(f^\delta)_i(x) - f_i^\delta(x)| + |f_i^\delta(x) - f_i(x)|$$

$$< \delta\ |f_i^\delta(x)| + \varepsilon \quad \text{by } 4^o \text{ and } 7^o.$$

$$\leq \delta D + \varepsilon$$

$$\leq 2\ \varepsilon \qquad \forall i$$

$8^0 \Longrightarrow f^\delta - f + 2\ \varepsilon\ u$ is non-decreasing where $u = \sum_{i=1}^{n} x_i$. Now define

$$h = f^{+\delta} - f^+(0) + 2\ \varepsilon\ u\ , \text{ and}$$

$$k = h-f = (f^+-f)^\delta + (f^\delta - f+2\ \varepsilon\ u) - f^{+\delta}(0)$$

Note that this h and k meet the demands of h and k stated at the end of 2^o above. Thus, $h o \lambda^k$ and $k o \lambda^k \varepsilon A^+$, Hence,

$$||v||_A \leq h(\underline{1}) + k(\underline{1}) = 2\ h(\underline{1}) - f(\underline{1}) \text{ and}$$

$$||v||_{BV} \leq ||v||_A \leq 2\ f^{+\delta}(\underline{1}) - 2\ f^{+\delta}(0) + 4\ \varepsilon\ u(\underline{1}) - f(\underline{1})$$

$$\leq 2\ f^+(\underline{1}) - 2f^+(0) + 4\ \varepsilon\ n - f(\underline{1}) \quad (\text{by } 5^o)$$

$$= f^+(\underline{1}) + f^-(\underline{1}) + 4\ \varepsilon\ n$$

$$= ||v|| + 4\ \varepsilon\ n.$$

Since ε is arbitrary lemma is proved. Q.E.D.

Lemma 1.24: Let $L = \{\mu \in NA : \mu << \lambda\}$, where λ is the Lebesgue measure. Then $L \subset \bar{A}$, the BV-closure of A.

Proof: Let $\mu \in L$. Then $\mu << \lambda$. So, by Radon-Nikodyn theorem, there exists $f \in L_1(\lambda)$ s.t.

$$\mu(S) = \int_S f\, dt \qquad \forall S \in \mathcal{B}.$$

Given $\varepsilon > 0$, there exists simple function $s = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$ s.t.

$$\int |f-s|\, dt < \varepsilon, \qquad \text{(see (5) Theorem 3.13).}$$

where $\{A_i\}_1^k$ is a partition of I and $\alpha_i \in \mathbb{R}$. It is also possible to choose A_i's as diadic intervals of I. Define,

$$\eta(S) = \int_S s(t)\, dt.$$

Note that $\eta \in A$ and $||\mu-\eta||_{BV} \leq \int |f-s|\, dt < \varepsilon$. Thus μ can be approximated by members of A and hence $\mu \in \bar{A}$.

Q.E.D.

Now, we turn to the proof of the proposition 1.19. Note the following observations:

1° Since A is a Banach algebra, so is $\bar{A}$ and hence lemma 1.24 implies that $\bar{A}$ contains all the polynomials in measures in L and also their limits. But the problem is that $L \neq$ whole of NA. For this

2° Let $v \in PNA$. Then v is the BV-limit of a sequence $\{p_n\}$ of polynomials in NA^1 measures. Since each p_n involves finitely many such measures, there are only countably many such measures involved in the entire sequence; let them be $\mu_1, \mu_2, \ldots\ldots$ Set

$$\mu = \sum_{i=1}^{\infty} \mu_i / 2^i$$

Note that for each i, $\mu_i << \mu$. Now, by proposition 1.16 there exists an automorphism θ of $(I, \mathcal{B})$ s.t.

$$\theta_* \mu = \lambda$$

Note that for such an automorphism $\theta_* p_n$ are polynomials in $\theta_* \mu_i$ all of which are $<< \theta_* \mu = \lambda$. Hence by 1° above $\theta_* p_n \in \bar{A}$ $\forall n$ and

$$||\theta_* p_n - \theta_* v|| = ||p_n - v|| \xrightarrow[n \to \infty]{} 0$$

$$\Longrightarrow \quad \theta_* v \in \bar{A} \quad \Longrightarrow \quad v \in \theta_*^{-1} \bar{A} .$$

Now exercise 1.18, lemma 1.23 and the fact that "for any internal space A, θ_*^{-1} A is also internal, where θ_* as defined above, imply that each member of pNA belongs to some internal subspace of pNA. Hence pNA is internal.

Q.E.D.

LECTURE 2

Axiomatic Value on pNA

2.1 Introduction

What will be discussed here is the generalisation of the value solution concept - originally defined for games with finite number of players by Shapley (1953) - to the games with continuum of players. We are not inclined to go deep inside the philosophical aspect of this generalization. To get an extensive account of it, readers should refer to Shapley and Shubik (6) : specifically chapter 2 and 10, Auman and Shapley (1) : general introduction, and sections 28, 29, 30. However, we can think of these games models with continuum of players as the limiting models for finite games in the same spirit as in physical sciences, we represent a large number of discrete particles that make up a fluid by a continuous medium. There are two ways through which one can pass from finite models to the infinite models - one is by replicating each type of agents and the other one is by fracturing each agent into parts. First one leads to the games with countable number of players; whereas -the other way leads to the game models with continuum of players.

Note that the continuum models could again be of different types. One possibility is non-atomic games where each player is insignificant - i.e. has weight zero in influencing the outcome of the game. Another possibility is continuous games with some players who have positive weights in influencing the outcome of the game together with a continuum of players of the first type. Another extreme possibility to the non-atomic games is the games where each player has positive weight in influencing the outcome. Our basic problem is : Suppose the "grand coalition" has learned something cooperatively. Now, problem is who will get how much? Different solution concepts are there in the literature like core, bargaining set, stable set, Shapley Value - each with its own ethical view-point behind. Here we shall restrict ourselves to only one solution concept, namely Shapley value. Mathematically, Shapley Value is, perhaps, most tractable of all other cooperative solutions; and it takes into account the power structure of different coalitions explicitly in determining the outcome.

For all finite games, Shapley has provided the existence of a unique value operator and a 'nice' formula for it. But for continuum analogue, there is no such general result. However, A-S have proved the existence of a unique value operator on quite a big subspace of games namely on bv'NA and have provided with a computational formula for it involving differential and integral calculus on pNA, an economically very important subclass of games in bv'NA.

Recently, generalising the concept of derivative J.F.Marten (7) has provided the existence of value and its computational formula for more inclusive space than bv'NA.

2.2 The axioms of the Value

Definition 2.1: A game with side payment is a set function.

Let $\mathcal{H}$ be the group of all automorphisms of $(I, \mathcal{C})$, (i.e., borel isomorphisms of $(I, \mathcal{C})$ onto itself). Each $\theta \in \mathcal{H}$ induces a linear map θ_* of BV onto itself, defined by

$$(\theta_* v)(S) = v(\theta S) \qquad \forall\, S \in \mathcal{C}.$$

Definition 2.2: A linear subspace Q of BV is said to be symmetric if

$\theta_* Q \subset Q \qquad \forall\, \theta \in \mathcal{H}$.

<u>Definition 2.3</u>: Let Q be a symmetric subspace of BV. The <u>axiomatic value on Q</u> is a mapping $\phi : Q \rightarrow FA$ satisfying the following axioms:

(E.2.1) ϕ is linear (linearity)

(E.2.2) $\phi(Q^+) \subset FA^+$ (positivity)

(E.2.3) $\phi \theta * = \theta * \phi \quad \forall \theta \in \mathcal{H}$ (symmetry.)

(E.2.4) $(\phi v)(I) = v(I)$ (efficiency)

[(E.2.5)* ϕ is a projection on FA (projection)]

Now let us briefly explain what these axioms say. 'Linearity' axiom says that if it is possible physically to play two independent games simultaneously by the same set of players then in the combined games they should get the sum of the pay off of the component games. 'Symmetry' axiom says that the value is not dependent on how the players are named. Or in otherwords, the symmetry axiom makes the value impresonal. 'Effiency' axiom makes sure that the value distributes away the total income to its players. 'Positivity' axiom says that if a player can contribute some positive amount to some coalition, then he should get a positive amount by the value allocation. Last axiom means that if the game already is additive, then it should also be the value allocation.

2.3 Existence of Value and its Formula on pNA.

First, let us note that there is no value on BV. For justification, consider the following:

<u>Example 2.4:</u> (Unanimity game): Let

$$v(S) = 1 \text{ if } S = I$$
$$= 0 \text{ otherwise.}$$

It is trivial to check that $v \in BV$. Suppose there exists a value ϕ on BV. Then $\mu = (\phi\ v) \in FA$.

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* Note * (E.2.5) is not needed for the present analysis. Later on we shall use it.

Claim : $\mu = 0$ identically. For, first note that v is invariant under all automorphisms. Hence μ is also by E.2.3. Symmetry axiom will imply that $\mu(\{i\}) = 0 \ \forall \ i \in I$ if not, then assume $\mu(\{i\}) = \delta > 0$. Symmetry axiom ensures that $\mu\{i\} = \mu\{j\}$. Let $n_o > \frac{1}{\delta}$ be an positive integer. Let S = set consisting of n_o elements from I. Then $\mu(S) = \sum_{i \in S} \mu(\{i\}) = n_o \delta > 1$. contradiction that μ is a finitely additive measure with $\mu(I) = 1$.

Now, let $S = [0, 1/2]$. The following automorphism

$$\theta(x) = \frac{1}{2} x \quad \text{if} \quad x \in [0, 1/2]$$
$$= x - \frac{1}{2}(x-1) \quad \text{if } x \in (\frac{1}{2}, 1]$$

we have $\theta\,[0, \frac{1}{2}] = [0, \frac{1}{4}]$. Hence $\mu([0, \frac{1}{2}]) = \mu([0, \frac{1}{4}])$. Similarly, one can show easily that $\mu([0, \frac{1}{4}]) = \mu([\frac{1}{4}, \frac{2}{4}])$. Hence $\mu(S) = \mu[0, \frac{1}{4}] + \mu[\frac{1}{4}, \frac{2}{4}] = 2\,\mu(S) \Longrightarrow \mu(S) = 0$. Similarly, $\mu([\frac{1}{2}, 1]) = 0$. Hence $\mu(I) = 0$. Contradiction to E.2.4.

Note that the game in example 2.4 is not non-atomic. So one may hope that all non-atomic games in BV have value. The following exercise gives the negative answer.

Exercise 2.5: Let

$$v(S) = 1 \quad \text{if } I-S \text{ is a finite set}$$
$$= 0 \quad \text{otherwise.}$$

Show that v is a non-atomic BV-game and there is no value for it.

But life is not so bad. There is some 'important' spaces on which value exists. One such space is what will be told now.

Let pNA be the closed subspace of BV generated by all powers of NA^+ measures. pNA is a symmetric subspace. A game of the form $f \circ \mu$ where $\mu \in (NA)^m$ and f is a real valued function on $R(\mu)$ with $f(0) = 0$, is called vector measure game. The following theorem assures the existence of value on pNA.

Theorem 2.6 (E, in A-S): (i) There exists a unique value ϕ on pNA. Further (ii) let $\mu \in (NA)^m$ and $f \in C^1(R(\mu))$ with $f(0) = 0$. Then $fo\mu \in pNA$ and

(E.2.6) $$\phi\,(fo\mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

where $R(\mu)$ is the range of μ and $f_{\mu(S)}$ directional derivative of f in the direction $\mu(S)$.

Remark 2.7: (1^R) Before proving this theorem, we have to establish a series of results. A complete characterisation of the vector measure games in pNA is done in lecture 4.

(2^R) The analogue of the formula E.2.6 for other general games in pNA will be done in the next lecture.

Proof of the Theorem 2.6:

[1^S] First we shall prove the first part of (ii) in several steps. 1st part of theorem (ii) is true if f is a polynomials in m variables.

Proof: Note the following identity : Let $k > 0$ be an integer and $x_1, \ldots, x_k \in \mathbb{R}$. Then,

(E.2.7) $$k!\; x_1, \ldots, x_k = (x_1+\ldots+x_k)^k - \sum_{1\le i\le k} (x_1+\ldots+x_k-x_i)^k + \sum\sum_{1\le i<j\le k} (x_1+x_2+\ldots+x_k - (x_i+x_j))^k - \ldots$$

Now let $fo\mu = \Sigma\, a_{i_1\ldots i_m}\, \mu_1^{i_1}\ldots\mu_m^{i_m}$, $\mu \in (NA)^m$ and $f(0) = 0$. By E.2.7 and the fact that any $\mu \in NA$ can be decomposed into $\mu = \mu^+ - \mu^-$, both $\mu^+, \mu^- \in NA^+$, (E.g. Hahn decomposition) it follows that $fo\mu \in pNA$.

[2^S] The polynomials are dense in $C^1(R(\mu))$ w.r.t. $||\cdot||_1$ norm as defined below. w.l.g. (for justification see A-S pp.42) we assume $\dim R(\mu) = m$ for $\mu \in (NA)^m$ and $||f||_o = \max_{x\in R(\mu)} |f(x)|$, define

$$||f||_1 = ||f||_o + \sum_{i=1}^m ||f_i||_o,$$

where $f_i(x)$ in the definition of $||f_i||_o$ is the continuous extension of the partial derivatives f_i from the interior of $R(\mu)$ to its boundary.

Proof: See (8) p.68. They show that any $f \in C^k(R)$, where R is an n-dimensional cube can be approximated in $||.||_o$ by a sequence of polynomials $\{p_n\}$ s.t.

$$\frac{\partial^r p_n}{\partial x_i \ldots \partial x_j} \xrightarrow{||\ ||_o} \frac{\partial f}{\partial x_i \ldots \partial x_j} \quad \forall\ r \leq k \text{ and } i,j \leq m.$$

So, in particular, extending our f form $R(\mu)$ which is a compact convex subset of $\mathbb{R}^m$ to a cube in an arbitrary way, we establish 2^S.

[3^S] Let us fix a $\mu \in (NA)^m$ arbitrarily. Then $||\ .||_1$-convergence of set functions of the form $f o \mu$, $f \in C^1(R(\mu))$ with $f(0) = 0 \implies ||.||_{BV}$ convergence i.e. $\exists\, c > 0$ s.t. $||.||_{BV} < c||.||_1$.

Proof: Let ∇F be the gradient vector of f in the interior of $R(\mu)$. Consider a chain $\Omega : \phi = S_o \subset S_1 \subset \ldots \subset S_n = I$ s.t. $\mu(S_i) \in \text{Int } R(\mu)\ \forall i$ except possibly the first and the last. Now,

$$||v||_\Omega = \sum_{j=o}^{m} |\ f o \mu\ (S_{j+1}) - f o \mu\ (S_j)|$$

$$= \sum_{j=o}^{n} \mu(S_{j+1}-S_j)\ \nabla F(\mu(S_j) + \theta_j\ \mu(S_{j+1}-S_j)) \text{ for some}$$

$$0 \leq \theta_j \leq 1 \quad \forall\ j \quad \text{(By mean value theorem)}$$

$$\leq \sum_{j=0}^{n} \sum_{i=1}^{m} \mu_i(S_{j+1}-S_j)||f||_1 \leq ||f||_1 \sum_{i=1}^{m} \mu_i(I)$$

$$\leq ||f||_1 \sum_{i=1}^{m} ||\mu_i||, \quad \forall \Omega$$

Take $c = \sum_{i=1}^{m} ||\mu_i||$ and appeal to proposition 1.4 completes 3^S.

[4^S] 1st part of (ii) of the theorem is true when f is of the form $f o \mu$; $f \in C^1(R(\mu))$, $\mu \in (NA)^m$ with $f(0) = 0$. Because of 2^S, $f o \mu$ is the

$||.||_1$-limit point of a sequence of polynomials in m-variables. Hence it is the $||.||_{BV}$ limit point of the sequence andhence fo $\mu \varepsilon$ pNA.

<u>Definition 2.8</u>: Let $\mu \varepsilon (NA)^m$. A <u>neighbourhood in $R(\mu)$ of the diagonal $[0,\mu(I)]$</u> is the set

$$N(\mu,\varepsilon) = \{\mu(S) : || \mu(S) - t\,\mu(I)|| \leq \varepsilon \text{ for some } t \varepsilon [0,1],\ S \varepsilon \mathcal{B}\ \forall\varepsilon>0\}$$

Let $Q = \{v \varepsilon \text{ pNA}: \exists\, m>0,\ \mu\varepsilon(NA)^m,\ f\varepsilon\, C^1(R(\mu)) \text{ with } f(0) = 0 \text{ and } N(\mu,\varepsilon)\ \varepsilon>0 \text{ s.t.}$
$v(S) = f o\mu(S) \text{ whenever } \mu(S)\ \varepsilon\ N(\mu,\varepsilon)\}$

For $v \varepsilon Q$, define

(E.2.8) $\phi(v)(S) = \int f_{\mu(S)}(t\mu(I))dt,$

where μ and f are as in the definition of v in Q, which are not necessarily unique.

$[5^S]$ ϕv is countably additive. ϕ is a well defined linear operator on Q with $||\phi(v)|| \leq ||v||$. In fact $||\phi||= 1$ on Q.

<u>Proof</u>: See A-S pp. 44-46.

$[6^S]$ Q is a dense subspace of pNA.

<u>Proof</u>: Suppose v_1, v_2 ε Q. To show v_1+v_2 ε Q. Now, v_i ε Q ====> there exist $m_i > 0$, $\mu_i\ \varepsilon(NA)^{m_i}$, $f_i\ \varepsilon\ C^1(R(\mu_i))$ and $N_i(\mu_i,\varepsilon_i)$, $\varepsilon_i > 0$ s.t.
$S\ \varepsilon\ N(\mu_i,\varepsilon_i)$ ===> $v_i(S) = f_i o\ \mu_i(S)$, $\forall i=1,2$,

Define $\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$ $f = goh$ where $h = \begin{pmatrix}f_1\\ f_2\end{pmatrix}$ $g = f_1+f_2$,

$$N \subset N_1(\mu_1, \varepsilon_1) \times N_2(\mu_2, \varepsilon_2).$$

It is now trivial that v_1+v_2 ε Q. It's trivial to show that $\alpha \varepsilon \mathbb{R}$ $v \varepsilon Q$ ===> $\alpha v \varepsilon Q$. Now note that Q contains all powers of NA measures hence 6^S is proved.

$[7^S]$ Since $||\phi||= 1$ on Q, a dense subspace of pNA and since by proposition 1.7 FA is complete, ϕ can be extended uniquely to $\bar{Q}$ = pNA, preserving the norm.

[8^S] Let ϕ : pNA$\rightarrow$ FA be a linear map satisfying:

(i) $||\phi|| \leq 1$

(ii) $\phi(\mu^n) = \mu \quad \forall \ \mu \in NA^1$

Then ϕ is a value on pNA.

Proof: $(8.1)^o$: (E.2.3): Let $f(x) = x^n$ and $v = f \circ \mu$, $\mu \in NA^1$. Then for each $\theta \in \Theta$, $\theta_* \mu \in NA^1$. Hence

$$\phi\theta_* v = \phi(f \circ \theta_* \mu) = \theta_* \mu = \theta_* \phi v.$$

Now since both θ_* and ϕ are continuous on pNA and since $\phi\theta_* - \theta_*\phi = 0$ for all games in the spanning class hence also equal to zero on pNA. Thus ϕ satisfies (E.2.3).

$(8.2)^o$: (E.2.4): The mapping $v \mapsto (\phi v(I) - v(I))$ is a continuous linear mapping that vanishes on the spanning class of pNA hence it vanishes identically on pNA. Hence E.2.4.

$(8.3)^c$ (E.2.2): Suppose v in pNA is monotonic and suppose ϕ is not positive then $\exists$ S $\in \mathcal{B}$ s.t. $(\phi v)(S) < 0$. Then

$$v(I) = ||v|| \geq ||\phi v|| \qquad \text{(by hypothesis (i))}$$

$$\geq |(\phi v)(S)| + |(\phi v)(I) - (\phi v)(S)|$$

$$\text{(Taking } \Omega : \phi \subset S \subset I)$$

$$> |v(I) - (\phi v)(S)| \qquad \text{(by } (8.2)^o \text{ above)}$$

$$= v(I) - (\phi v)(S) > v(I) \qquad \text{(contradiction!)}$$

Thus $v \in pNA^+ \Longrightarrow \phi v \in FA^+$.

$(8.4)^o$ (E.2.1): from 5^S . Hence 8^S.

[9^S] Observe that if $v = \mu^k$, $\mu \in NA^1$, then the formula E.2.8 yields

$$(\phi v)(S) = \mu(S) \int \frac{d}{dt}(t^k)dt = \mu(S).$$

Hence by 8^S above ϕ is indeed a value on pNA.

So it remains to prove the uniqueness. To prove it, let us observe the converse of 8^S.

[10^S] Let ϕ be a value on pNA. Then, $\phi(\mu^n) = \mu \ \forall \ \mu \in NA^1$.

Proof: Left as an exercise to the reader, see for generalised version A-S, p. 38, proposition 6.1.

[11^S] Let Q be an internal subspace of BV and let ϕ be a positive linear operator from Q into BV obeying the normalisation condition E.2.4 Then $||\phi|| \leq 1$

Proof: Suppose $v \in Q^+$.

$$||\phi v|| = \phi v(I) \qquad \text{Since } \phi \text{ is positive}$$
$$= v(I) \qquad \text{by E.2.4.}$$
$$= ||v|| \qquad \text{since } v \in Q^+$$

Now since Q is internal, for any $v \in Q$, and given $\varepsilon > 0$ there exist u and $w \in Q^+$ s.t.

$$||v|| + \varepsilon \geq ||u|| + ||w|| \qquad \text{and}$$

$$v = u - w.$$

Now, $||\phi v|| = ||\phi(u-w)|| \leq ||\phi u|| + ||\phi w|| = ||u|| + ||w|| \leq ||v|| + \varepsilon$

Since ε is arbitrary 11^S is proved.

12^S below completes the uniqueness proof.

[12^S] By linearity of ϕ and by 10^S one ensures the uniqueness of the value operator on the set of all polynomials in NA measures which is dense in pNA. Now 11^S say that ϕ is continuous on pNA. Hence it must be unique on the whole of pNA.

Q.E.D.

Remark 2.9: Let P = space of all polynomials in NA-measures. 1^S above implies that P is same as the space of all linear combinations of powers of NA^1-measures. Thus pNA = $\bar{P}$. It is much easier to prove the existence of unique value on P than on pNA. See A-S note 2, p.54.

LECTURE 3

Generalized Sets and Value formula for General Games in pNA

3.1 Introduction:

The value formula E.2.6 in the previous lecture holds for those games only in pNA for which there exists a non-atomic vector measure $\mu \in (NA)^m$ s.t. in some compact convex neighbourhood in $R(\mu)$ of the diagonal the game is representable by a continuously differentiable function. To provide formula for general games in pNA, Aumann and Shapley proceed by generalising the notion of set, what are called the ideal sets; the ordinary sets will be identified with some types of ideal sets. Then they extend the set functions from the σ-algebra of ordinary sets to the class of ideal sets and then they provide a formula (which is the extension of E.2.6 when properly identified) in terms of the directional derivatives of these extended set functions. Let us proceed now, systematically:

3.2 Ideal Set Functions on the Lattice of ideal Sets

Definition 3.1: An ideal set f is a measurable function from $(I, \mathcal{B})$ to $(I, \mathcal{B})$.

We can attach an intuitive meaning to these ideal sets as follows: an ideal set f specifies the degree, f(x), $0 \leq f(x) \leq 1$ with which the point x from I belongs to a set. An ordinary set $S \in \mathcal{B}$ is identified by χ_S. Let

$B(I, \mathcal{B})$ = space of all bounded real valued measurable functions on $(I, \mathcal{B})$

$B_1(I, \mathcal{B})$ = Space of all ideal sets.

$$\overline{\mathcal{B}} = \{\chi_S : S \in \mathcal{B}\}$$

Note that $B(I, \mathcal{B}) \supset B_1(I, \mathcal{B}) \supset \overline{\mathcal{B}}$.

Sometimes we will denote $B(I,\mathcal{B})$ and $B_1(I,\mathcal{B})$ simply by B and B_1 respectively. Note also that B is a linear space over the real field.

On B_1, let us define the following binary operations : for each $f,g, \in B_1$

$$f \leq g \Longleftrightarrow f(x) \leq g(x) \quad \forall x \in I$$

$$f \vee g = \max(f,g)$$

$$f \wedge g = \min(f,g)$$

It's very easy to check that B_1 with $\leq$ V, Λ is a distributive lattice but not complemented. Had it been complemented also then the set of ideal sets-B_1-would have been isomorphic to the Boolean algebra of all open closed subsets of a totally disconnected compact Hansdorff space by Stone's Representation Theorem; and hence nothing new could have been achieved. But $\bar{\mathcal{E}}$ is a distributive and complemented sub-lattice of it.

Let us define two more operations on B_1 whenever defined

$$(f + g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

for every $f,g \in B_1$ $\alpha \geq 0$ real number and $\forall x \in I$.

Let us denote by

$$1 = \chi_I$$

$$0 = \chi_\phi$$

So, we have achieved the following analogue:

On ordinary sets i.e. $\mathcal{B}$,	on ideal sets, i.e. B_1
A,B are the elements	f,g are the elements
ϕ	0
I	1
$\subseteq$	$\leq$
$\cup$	$\vee$
$\cap$	$\wedge$

Definition 3.2 An ideal set function is a maping $\bar{v}: B_1 \to \mathbb{R}$ with $\bar{v}(0) = 0$.

Definition 3.3. An ideal set function $\bar{v}$ is said to be monotonic if $\forall\, f,g \in B_1$ with $f \leq g \implies \bar{v}(f) \leq \bar{v}(g)$. $\bar{v}$ is said to be of bounded variation if there exist monotonic ideal set ;functions $\bar{u}$ and $\bar{w}$ s.t. $\bar{v} = \bar{u} - \bar{w}$. Let

IBV = set of all ideal set ;functions of bounded -variations.

Define, for each $\bar{v} \in$ IBV,

$$||\bar{v}||_{IBV} = \inf \{\bar{u}(\chi_I)+\bar{w}(\chi_I) : \bar{u} \text{ and } \bar{w} \text{ are monotonic ideal set functions and } \bar{v}=\bar{u}-\bar{w}\}$$

One, at once, varifies that $||.||_{IBV}$ is indeed a norm on IBV. In natural way, one can extend the definition of chain and many other concepts, originally developed for ordinary set functions, and prove the analogues of different results like propositions 1.4,1.6 etc.

We shall equip $B = B(I, \mathcal{B})$ with two topologies. Let

$$[NA] = \{\bar{\mu}(f) = \int f d\mu \ : \ f \in B \text{ and } \mu \in NA\}$$

Note that $\bar{\mu}$'s in [NA] are in fact linear functionals.

Definition3.4. The NA-topology on B is the [NA]-weak topology, i.e., the weakest topology on B such that all linear functionals in [NA] are continuous with respect to usual topology on $\mathbb{R}$. The DNA topology on B is the [NA]-weak topology w.r.t the discrete topology on $\mathbb{R}$.

In this lecture, we ;shall use only NA-topology unless otherwise stated; DNA-topology will be used in a later lecture for more general results due to Marten (7).

Proposition 3.5. $\bar{\mathcal{B}}$ is DNA-dense in B_1 and hence, in particular, it is NA-dense in B_1 also.

Proof: Let us first extend the non-atomic measures from $\mathcal{B}$ to B_1 in the obvious way as follows:

(E.3.1) $\bar{\mu}(f) = \int_I f d\mu$

Note that $\bar{\mu}(\chi_S) = \mu(S)$, $\forall$ S $\varepsilon\, \mathcal{B}$. Fix f ε B_1. Note that $0 \leq \bar{\mu}(f) \leq 1$, hence by Lyapunov's theorem there exists S $\varepsilon\, \mathcal{B}$ s.t. $\mu(S) = \mu(\chi_S) = \bar{\mu}(f)$. Now, sub-basis elements for DNA-topology are

$$N(f) = \bar{\mu}(\bar{\mu}^{-1}(f)) = \{g \in B_1 : \bar{\mu}(g) = \bar{\mu}(f)\},\ \forall f \in B_1, \mu \in NA$$

Note that N(f) contains a $\mu\chi_S$ for some S $\varepsilon\, \mathcal{B}$.

Q.E.D.

(E.3.1) above is the extension of a measure from $\mathcal{B}$ to B_1. Now, the following is an extension for general set functions. Or in otherwords, the following is the extension of the definition E.3.1 from set of non-atomic measures to the set of all set functions.

<u>Definition 3.6.</u> Let Q be a linear subspace of BV. The following mapping "___" from Q to IBV satisfying following will be called <u>an extension</u>:

(E.3.2) $\overline{(\alpha u + \beta w)} = \alpha\bar{u} + \beta\bar{w}$

(E.3.3) $\overline{v\, w} = \bar{v} \cdot \bar{w}$

(E.3.1) $\bar{\mu}(f) = \int f\, d\,\mu$

(E.3.4) $u \in Q^+ \Longrightarrow \bar{u} \in IBV^+$

whenever u,w ε Q, $\alpha,\beta \in \mathbb{R}$, μ ε NA and f ε B_1. The ideal set function $\bar{v}$ is called the <u>extension of v.</u> Note that if Q is an algebra, Then '___' is a homomorphical imbedding of Q into IBV with extra properties that measures are identified with 'extended measures' in IBV and monotonic set functions are identified with extended montonic set functions in IBV.

Let us introduce another non-topological concept in B_1:

Definition 3.7: A real valued function, w, defined on B_1 or on any subset of B_1 is said to be uniformly continuous if, for given $\varepsilon > 0$, there exist a vector μ of NA measures and a $\delta > 0$, s.t. $\forall\ f,g \in dom(w)$,

$$||\int(f-g)d\mu|| < \delta \Longrightarrow |w(f)-w(g)| < \varepsilon,$$

where $||.||$ above is the 'max' norm in $\mathbb{R}^n$, n= no. of components in μ .

Theorem 3.8: (Theorem G of A-S). There exists a -unique extension for Q=pNA.

Proof: We shall prove it ;in several steps.

$[1^S]$ Each v in pNA is uniformly continuous on $\mathcal{B}$.

Proof: $(1.1)^S$ If $v \in NA$ then v is uniformly continuous on $\mathcal{B}$, which follows from definition of uniform continuity.

$(1.2)^S$ If u and w are uniformly continuous so are u+w and u,w.

Now 1.1^S, 1.2^S together with E.3.2 and E.3.3 imply that all polynomials in NA-measures, are uniformly continuous.

$(1.3)^S$ To show now that any general game, v, in pNA is u.c. (uniformly continuous): Since polynomials in NA measures are dense in pNA given $\varepsilon > 0$, $\exists\ v_\varepsilon$, a polynomial in NA measures s.t.

$$||v - v_\varepsilon|| \leq \varepsilon$$

For any two $S,T \in \mathcal{B}$, we have

$$|(v-v_\varepsilon)(S) - (v-v_\varepsilon)(T)| \leq |(v-v_\varepsilon)(S)| + |(v-v_\varepsilon)(T)| \qquad \text{(E.3.5)}$$

$$\leq ||v-v_\varepsilon|| + ||v-v_\varepsilon||$$

$$\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Now since v_ε is u.c. on $\mathcal{B}$, there exists a vector of NA-measures and a $\delta > 0$ s.t. $\forall\ S,T \in \mathcal{B}$

$$|\mu(S) - \mu(T)| < \delta \Longrightarrow |v_\varepsilon(S) - v\ (T)| < \varepsilon/2. \qquad \text{(E.3.6)}$$

By (E.3.6) and (E.3.5),

$$|\mu(S) - \mu(T)| < \delta \implies |v(S) - v(T)| \leq |(v-v_\varepsilon)(S) - (v-v_\varepsilon)(T)| + |v_\varepsilon(S) - v_\varepsilon(T)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This completes the proof of 1^S.

$[2^S]$ Let $v \in pNA$. We extend v from $\mathcal{B}$ to a uniformly continuous function $\bar{v}$ on B_1 in the following way:

Let $\varepsilon_n = 1/n$. Let $\mu^{(n)}$ and δ_n be respectively the vector of measures and δ's corresponding to ε_n in the definition of uniform continuity of v. Let $g \in B_1$. We want to define $\bar{v}(g)$. We can choose (because of proposition 3.5), for each n, $S_n = S_n^g \in \mathcal{B}$ such that

$$||\int(\chi_{S_n} - g)d\,\mu^{(n)}|| \leq \delta_n/3$$

$(2.1)^S$: $\{v(S_n)\}$ is a Cauchy sequence. To prove it we can assume w.l.g. (How?) that $\delta_n \downarrow$ and for $m \geq n$ all the co-ordinates of $\mu^{(n)}$ are also the co-ordinates of $\mu^{(m)}$ (Thus in general, the dimension of $\mu^{(m)}$ is $\geq$ dimension of $\mu^{(n)}$). From this it follows that for $m \geq n$

$$||\int(\chi_{S_m} - g)d\,\mu^{(n)}|| \leq ||\int(\chi_{S_m} - g)d\,\mu^{(m)}|| < \frac{\delta_m}{3} \leq \frac{\delta_n}{3}.$$

Hence,

$$||\mu^{(n)}(S_m) - \mu^{(n)}(S_n)|| = ||\int(\chi_{S_m} - \chi_{S_n})d\,\mu^{(n)}|| < \frac{2\delta_n}{3} < \delta_n$$

$$\implies |v(S_m) - v(S_n)| < \frac{1}{n} \quad \forall n \text{ by u.c. of } v \text{ on } \mathcal{B}. \qquad (*)$$

Hence $(2.1)^S$.

Define

$$\text{(E.3.7)} \qquad \bar{v}(g) = \lim_{n\to\infty} v(S_n^g)$$

$(2.2)^S$: $\bar{v}$ defined in (E.3.7) is u.c. on $\underline{B_1}$.

<u>Proof</u>: Fix n s.t. $\frac{1}{n} < \varepsilon/2$. Set $\delta = \frac{\delta_n}{3}$, $\mu = \mu^{(n)}$. Suppose

$||\int(f-g)d\mu|| < \delta$ We must show that

$|\bar{v}(f) - \bar{v}(g)| \not< \varepsilon$

In (*) above, let $m \to \infty$, we get

(E.3.8) $|\bar{v}(g) - v(S_n^g)| < 1/n < \varepsilon/3$

Similarly,

(E.3.9) $|\bar{v}(f) - v(S_n^f)| < \varepsilon/3$

On the otherhand,

$$||\mu(S_n^f) - \mu(S_n^g)|| \le ||\int(f - \chi_{S_n^f})d\mu|| + ||\int(g - \chi_{S_n^g})d\mu|| + ||\int(f-g)d\mu|| < \frac{3\delta_n}{3} = \delta_n.$$

Therefore,

(E.3.10) $|v(S_n^f) - v(S_n^g)| < 1/n < \varepsilon/3$

(E.3.8), (E.3.9), and (E.3.10) ====> $|\bar{v}(f) - \bar{v}(g)| < \varepsilon$ hence $(2.2)^S$.

$(2.2)^S$ ==> $\bar{v}$ is continuous on B_1 w.r.t. NA-topology.

$[3^S]$ $\bar{v}$ defined in 2^S is an extension of v, i.e. the mapping $v \to \bar{v}$ from pNA to IBV satisfies all (E.3.1) through (E.3.4). To prove these first note that

$$\bar{v}(\chi_S) = v(S) \quad \forall S \varepsilon \mathcal{C}$$

Now $\overline{(\alpha v + \beta w)} - (\alpha\bar{v} + \beta\bar{w})$, $\overline{vw} - \bar{v}\bar{w}$ and $\mu - \int_I(.)d\mu$ are all continuous extensions of 0 and hence must vanish identically. It remains to establish (E.3.4). To establish it, suppose $g_1, g_2 \varepsilon B_1$ and $g_2 \ge g_1$, and let $v \varepsilon (pNA)^{++}$. Since $\bar{v}$ is u.c., given $\varepsilon > 0$, $\exists\, \delta > 0$ and some vector of NA measures μ s.t.

$$||\int(f-g)d\mu|| < \delta \text{ ====> } |\bar{v}(f) - \bar{v}(g)| < \varepsilon.$$

Now, substitute $f = \chi_{T_i}$ and $g = g_i$. Then

$$||\int(f-g)d\mu|| = 0.$$

Since by Lyapunov's Theorem, $g_2 \geq g_1$ ===> there exist T_1, T_2, $T_2 \supset T_1$ s.t.

$$\mu(T_i) = \int g_i \, d\mu \quad , i=1,2.$$

Hence

$$|\bar{v}(\chi_{T_i}) - \bar{v}(g_i)| = |v(T_i) - \bar{v}(g_i)| < \varepsilon \quad , i = 1,2.$$

Now

$$|v(T_1) - \bar{v}(g_1) - v(T_2) + \bar{v}(g_2)| \leq 2\varepsilon$$

Since, v is monotonic, and $T_2 \supset T_1$, we have

$$\bar{v}(g_2) - \bar{v}(g_1) \geq -2\varepsilon \quad , \varepsilon \text{ is arbitrary.} \quad \text{Hence (E.3.4)}$$

Now, we shall prove the uniqueness of such an extension.

[4^S] The mapping '___' satisfying E.3.1 through E.3.4 on pNA is unique:

To prove it:

$(4.1)^S$: (E.3.1) ====> for $v \in NA$, $\bar{v}$ is unique, which together with (E.3.2) & (E.3.3) imply the uniqueness of the extension map '___' for all polynomials in NA measures. Now, it is enough to show:

$(4.2)^S$: The map '___': BV $\rightarrow$ IBV is continuous.

Proof: 1st note that $\bar{v}(\chi_I)/v(I) < K$ for some K, $\forall\, v \in (pNA)^+$. Now, since pNA is internal, for any $v \in$ pNA, there exist monotonic u and w s.t. $v = u-w$ and $||v|| \geq \dfrac{u(I) + w(I)}{2}$.

Now,

$$||\bar{v}|| = ||\bar{u}-\bar{w}|| \leq ||\bar{u}|| + ||\bar{w}|| = \bar{u}(\chi_I)+\bar{w}(\chi_I) \quad \text{by (E.3.4)}$$

$$\leq 2K \left(\frac{u(I)+w(I)}{2}\right) \leq 2K||v|| \qquad \text{Hence } (4.2)^S.$$

Q.E.D.

The following corollaries are the by products of the proof of the above theorem.

Corollary 3.9: $\bar{v}(\chi_S) = v(S) \quad \forall\, S \in \mathcal{B}$.

Proof: Take $S_n = S \;\; \forall n$ in (E.3.7)

Corollary 3.10: The extension map $v \mapsto \bar{v}$ is a continuous map from $(pNA, ||\cdot||_{BV})$ to $(IBV, ||\cdot||_{IBV})$.

Proof: See 4^S.

Corollary 3.11: $||\bar{v}||_{IBV} = ||v||_{BV}$.

Proof: Suppose $v \in pNA$. Since pNA is internal, given $\varepsilon > 0$, there exists $u, w \in (pNA)^+$ s.t.

$$v = u - w$$

and $\quad u(I) + w(I) \leq ||v|| + \varepsilon$

(E.3.4) ===> $\bar{u}$ and $\bar{w}$ are monotonic and (E.3.2) ===> $\bar{v} = \bar{u} - \bar{w}$. Now,

$$||\bar{v}||_{IBV} = ||\bar{u}-\bar{w}||_{IBV} \leq ||\bar{u}||_{IBV} + ||\bar{w}||_{IBV} = \bar{u}(\chi_I) + \bar{w}(\chi_I)$$

$$= u(I) + w(I) \leq ||v|| + \varepsilon.$$

Since ε is arbitrary and since $||\bar{v}||_{IBV} \geq ||v||_{BV}$ always, we have established 3.11.

Corollary 3.12: $\bar{v}$ is continuous on B_1 w.r.t. NA-topology. This follows from uniform continuity of $\bar{v}$ on B_1.

Remark 3.13: (1) Thus the extension mapping is an isometric isomorphism of the Banach algebra into the Banach algebra IBV with further property that it is positive and sending measures to extended measures.

(2) Extension possibilities for other more inclusive spaces than pNA should be also studied, though we do not need it at present.

Exercise 3.14: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then show that

$$\overline{f \circ v} = f \circ \bar{v} \qquad \forall\, v \in pNA.$$

Let us denote by

(E.3.11) $$\partial\bar{v}(t,S) = \frac{d}{d\tau}\bar{v}(t\chi_I + \tau\chi_S),$$

where $t \in [0,1)$ and the derivative is at $\tau = 0$. So $\partial\bar{v}(t,S)$ is the directional derivative of $\bar{v}$ at $t\chi_I$ in the direction χ_S. Note that for $v = f\circ\mu$, where μ is a vector of NA measures and $f \in C^1(R(\mu))$ with $f(0) = 0$.

$$\partial\bar{v}(t,S) = f_{\mu(S)}(t,\mu(I)).$$

The following theorem ensures the existence of above derivative for and generalises the value formula to all games in pNA:

Theorem 3.15: (Theorem H of A-S) For each v in pNA and each $S \in \mathcal{B}$, the directional derivatives $\partial\bar{v}(t,S)$ exists a.e.(λ) and belong to $L_1(\lambda)$ and if ϕ is the axiomatic value on pNA then

(E.3.12) $$(\phi v)(S) = \int_I \partial\bar{v}(t,S)d\,\lambda(t),$$

where λ is the Lebesgue measure on $(I,\mathcal{B})$.

Proof: Define, for each v,

$$E_v(t) = \overline{\lim_{\alpha\to 0}}\frac{\bar{v}(t\chi_I + \alpha\chi_S) - \bar{v}(t\chi_I)}{\alpha} - \underline{\lim_{\alpha\to 0}}\frac{\bar{v}(t\chi_I + \alpha\chi_S)-\bar{v}(t\chi_I)}{\alpha}$$

We have to show that $E_v(t) = 0$ a.e.t(λ) and $\partial\bar{v}(t,S) \in L_1(\lambda)$. To that end, let us fix $S \in \mathcal{B}$, and define for each $v \in$ pNA

$$||\bar{v}(t,S)||^+ = \overline{\lim_{\alpha\to 0}}\frac{\bar{v}(t\chi_I + \alpha\chi_S) - \bar{v}(t\chi_I)}{\alpha}$$

Lemma 3.16: $|\partial\bar{v}(t,S)|^+ \in L_1(\lambda)$ and $\int_I |\bar{v}(t,S)|^+ d\lambda(t) \leq ||v||$

Proof We shall prove it in several steps:

$(3.16.1)^S$: $|\partial\bar{v}(t,S)|^+$ is measurable, for, by Corollary 3.12, $\bar{v}(t\chi_I + \alpha\chi_S)$ is continuous in α. So, in the definition of $|\partial\bar{v}(t,S)|^+$ one can let α vary over rationals and hence $(3.16.1)^S$.

$(3.16.2)^S$: Lemma is true for all monotonic v's. To prove it observe that since v monotonic ===> $\bar{v}$ is also monotonic,

$$0 \leq \frac{\bar{v}(t\chi_I + \alpha\chi_S) - \bar{v}(t\chi_I)}{\alpha} \leq \frac{\bar{v}((t+\alpha)\chi_I) - \bar{v}(t\chi_I)}{\alpha}$$

Now, let $g(t) = \bar{v}(t\chi_I)$. So $g(t)$ is monotonic in t and hence a.e. differentiable. So

$$0 \leq \overline{\lim_{\alpha \to 0}} \frac{\bar{v}(t\chi_I + \alpha\chi_S) - \bar{v}(t\chi_I)}{\alpha} \leq \overline{\lim_{\alpha \to 0}} \frac{g(t+\alpha)-g(t)}{\alpha}$$

$$= g'(t) \text{ a.e.t}$$

i.e. $0 \leq |\partial\bar{v}(t,S)|^+ \leq g(t)$ a.e.t

Now $\int g'(t)dt \leq g(1)-g(0) = v(I) = ||\cdot||$ by the monotonicity of g(t).

Hence $(3.16.2)^S$.

$(3.16.3)^S$: We shall now prove the lemma for general v in pNA. Given $\varepsilon > 0$, there exist $u,w \in (pNA)^+$ s.t. $v = u-w$ and $||v|| + \varepsilon \geq ||u|| + ||w||$. By theorem 3.8 $\bar{v} = \bar{u}-\bar{w}$, and hence

$$|\partial\bar{v}(t,S)|^+ \leq |\partial\bar{u}(t,S)|^+ + |\partial\bar{w}(t,S)|^+$$

So by $(3.16.2)^S$,

$$\int|\partial\bar{v}(t,S)|^+ \, d\lambda(t) \leq ||u|| + ||w|| \leq ||v|| + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the proof of the lemma 3.16 is complete.

To continue the proof of the theorem 3.15, note that

$$0 \leq E_v(t) \leq 2\,|\partial\bar{v}(t,S)|^+.$$

Let

$$E_v = \int_I E_v(t)\, d\,\lambda(t)$$

E_v satisfies the following:

(i) $E_v \leq 2 \;||v||$

(ii) $E_{v+w} \leq E_v + E_w$, since $E_{v+w}(t) \leq E_v(t) + E_w(t)$

(iii) $E_v = 0 \;\; \forall \; v \;\; \varepsilon$ pNA.

To establish (iii) first note that $E_v = 0$ if $v = \mu^n$, $\mu \varepsilon$ NA. and hence by (ii) above $E_v = 0$ for all polynomials in NA measures. To show for general v in pNA, let, for given $\varepsilon > 0$, v_ε be a polynomial in NA measures s.t.

$$|| v-v_\varepsilon || < \varepsilon.$$

Now by (ii) and (i) above $E_v \leq E_v + E_{v-v_\varepsilon} \leq E_{v-v_\varepsilon}$

$$\leq 2||v-v_\varepsilon|| \leq 2\,\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, (iii) is established.

Now (iii) implies $E_v(t) = 0$ a.e. $t(\lambda)$ and hence $\partial\bar{v}(t,S)$ exists a.e. and whenever it exists, we have

$$|\partial\bar{v}(t,S)| = |\partial\bar{v}(t,S)|^+$$

and hence by lemma 3.16 above we have

$\int_I |\partial\bar{v}(t,S)| d\,\lambda(t) \leq ||v||$ and hence $\partial\bar{v}(t,S) \varepsilon L_1(\lambda)$

Now, we shall deduce (E.3.12):

For each $S \;\varepsilon\, \mathcal{C}$, define the mapping ψ_S: pNA $\to \mathbb{R}$ by

$$\psi_S\, v = \int \partial\bar{v}(t,S)dt$$

Note that ψ_S is a bounded linear functional on pNA and

$\psi_S v - \phi_S v = 0 \quad \forall\; v = f\circ\mu$, s.t. μ is a vector of NA measures and $f \;\varepsilon\; C^1(R(\mu))$ with $f(0) = 0$, where $\phi_S v = (\phi v)(S)$. Hence, in particular $\psi_S - \phi_S = 0$ for all polynomials in NA measures. Now, since $\psi_S - \phi_S$ is continuous and the set of polynomials in NA measure are dense in pNA, we have $\psi_S - \phi_S$ is identically 0 for all games in pNA and $\forall$ S.

Q.E.D.

Remark 3.17: Note that if $\bar{v}$ in IBV has a Fréchet derivative $D\bar{v}$ at $t\chi_I$ for some $t \in (0,1)$ then

$$\partial\bar{v}(t,S) = \frac{d}{d\alpha}\bar{v}(t\chi_I + \alpha\chi_S)\big|_{\alpha=0}$$

$$= \langle D\bar{v}(t, \chi_I), \chi_S\rangle ,$$

where $D\bar{v}(t\chi_I)$ is a bounded linear functional on Banach space $B(I, \mathcal{B})$ with sup norm and $\langle D\bar{v}(t\chi_I), \chi_S\rangle$ is its value at χ_S.

(E.3.13) shows that for that t for which $D\bar{v}$ exists, $\partial\bar{v}(t,S)$ is a measure on $\mathcal{B}$. Now the natural questions that come to one's mind are: For which $\bar{v}$'s the Fréchet derivative exists? Does there exist t in (0,1) s.t. $\partial\bar{v}(t,S)$ exists for all $S \in \mathcal{B}$? If yes, then these t's constitute how much of I and for which $\bar{v}$'s? Some partial answers are summarised in the following :

Proposition 3.18: Let $v \in$ pNA. Then for all most all $t \in (0,1)$, the following hold

(i) the extension $\bar{v}$ has a Frechet derivative $D\bar{v}$ at $t\chi_I$

(ii) $\partial\bar{v}(t,S)$ exists for all $S \in \mathcal{B}$

(iii) Let $\partial_t\bar{v}(S) = \partial\bar{v}(t,S)$. Then $\partial_t\bar{v}$ is a NA measure, and for all $f \in B_1$.

$$\langle D\bar{v}(t\chi_I), f\rangle = \int_I f \, d\,\partial_t\bar{v}$$

Proof: See A-S p. 157.

LECTURE 4

Characterisation of some games in pNA

4.1 Introduction

pNA is a very important subspace of BV from the view point of economic applications as well as that of having many strong results like formula for value in terms of differentials and integrals of certain functions. Under fairly general conditions, the set functions derived from exchange economies with transferable utilities [for non transferable utility models, same theory goes through by aggregating individual utilities by some weighting factors $\lambda(t)$, $t \in I$ and the corresponding value is called λ-value. For more details see (9)] or from production economies, both with the continuum of traders, have very interesting properties like (i) they are in pNA, (ii) there is a unique point in the core of these games and this unique point coincides with the value; this point is also competitive equilibrium.

Aumann and Shapley have given a nice value formula (E.2.6) for vector measure games $f \circ \mu$ in pNA. In fact, the games of this form where f is continuous, concave and homogeneous of degree 1 on $R(\mu)$ arise from market games of finite type, where the utility functions are not necessarily differentiable (See (11)). So, it is important to investigate what games of the form $f \circ \mu$ are in pNA. Theorem 4.2 will give a complete characterisation.

4.2 Some notations and concepts

$(NA)^m$ = linear space of m-vector of NA measures.

Define the norm on $(NA)^m$ by, for each $\mu \in (NA)^m$ $||\mu||_m = \sum_{i=1}^{m} ||\mu_i||_{BV}$, where $\mu = (\mu_1, \ldots \mu_m)$. Define $B(\mu)$ and $B(\mu,\varepsilon)$ for each $\varepsilon > 0$ by

$$B(\mu) = \{\eta \in (NA)^m : R(\eta) = R(\mu)\}$$

$$B(\mu,\varepsilon)=\{\eta \in (NA)^m : ||\mu-\eta||_m < \varepsilon\}$$

Fix a $\mu \in (NA^1)^m$ and a real function f on $R(\mu)$ with $f(0) = 0$. Define an operator

$$T_f : (B(\mu), ||.||_m) \to (BV, ||.||_{BV}) \quad \text{by}$$

$$T_f(\eta) = f\circ\eta \quad \forall\, \eta \in B(\mu).$$

Definition 4.1. f is said to be continuous at μ if T_f is continuous at μ

Theorem 4.2: (Tauman): Let $\mu \in (NA^1)^m$ and $f : R(\mu) \to$ with $f(0) = 0$. Then

$$f\circ\mu \in pNA \iff f \text{ is continuous at } \mu .$$

Proof: We shall give here only the idea of the proof. For details, see (10).

[<===]: Suppose f is continuous at μ. Assume $R(\mu)$ is of full dimension. Different steps are as follows: First smooth $f(x)$ by averaging it over a small cube $\mathcal{C}$ of volume a^m, very near to the point x in $R(\mu)$, for each $x \in R(\mu)$ in the following way:

For each $0 < \delta < 1$ and $x \in \chi R(\mu)$.

$$\text{(E.4.1)} \qquad f^\delta(x) = \frac{1}{a^m} \int_{y\in\mathcal{C}} f((1-\delta)x + \delta y)\, d\,\lambda(y) - \frac{1}{a^m} \int_{y\in\mathcal{C}} f(\delta y)\, d\lambda(y)$$

where λ is the Lebesgue measure on $\mathbb{R}^m$. Compare with $g^\delta(x)$ in the proof of proposition 1.19. One has to varify that the above function $f^\delta(x)$ is well-defined. Next step consists of proving that $f^\delta \in C^1(R(\mu))$. Same type of argument as used in proving g^δ belongs to $C^1(R(\mu))$ in the proof of proposition 1.19. Once, it is proved, theorem 2.6(ii) implies that $f^\delta\circ\mu \in$ pNA. Now, f is continuous at μ and some other results cooked from this hypothesis enable one to approximate $f\circ\mu$ by $f^\delta\circ\mu$ as $\delta \to 0$ in BV norm. Now, since pNA is closed in BV norm, $f\circ\mu \in$ pNA.

[===>] part: Suppose for some $\nu \in (NA)^m$, $f o \mu \in pNA$. Purpose is to show that f is continuous at μ. Different steps are:

$(4.2.1)^{\circ}$ $f o \mu \in pNA$ ===> there exists a $\{p_n\}$ of polynomials in NA^1 measures, say μ^n, s.t.

$$||p_n \circ \mu_n - f o \mu||_{BV} \to 0 \text{ as } n \to \infty$$

Define

$$\hat{R}_n = \{x \in R(\mu, \mu^n) : t_1(x) \in c\} ,$$

where $t_1(x)$ is the projection of x on $R(\mu)$ and $\mathcal{C}$ is as defined in the first part. On $\hat{R}_n$, choose a measure λ_n s.t. for each $0 < \delta < 1$.

$$f_n^{\delta}(x) = \int_{y \in \hat{R}_n} f((1-\delta)x + \delta t_1(y)) \, d\lambda_n(y) - \int_{y \in \hat{R}_n} f(\delta t_1(y) d\lambda_n(y)$$

defined on $R(\mu)$ is independent of n(this is possible!) Set $f^{\delta} = f_n^{\delta}$.

For each $0 < \delta < 1$, note that $f^{\delta} \in C^1(R(\mu))$. Similarly, for each n and $0 < \delta < 1$, we defined p_n^{δ} on $R(\mu_n)$ by

$$p_n^{\delta}(x) = \int_{y \in \hat{R}_n} p_n((1-\delta)x + \delta t_2(y)) d\lambda_n(y) - \int_{y \in \hat{R}_n} p_n(\delta t_2(y)) d\delta_n(y)$$

where $t_2(y)$ is the projection of y on $R(\mu^n)$.

$(4.2.2)^{\circ}$: $||p_n^{\delta} \circ \mu^n - f^{\delta} o \mu||_{BV} \to 0$ uniformly in $0 < \delta < 1$.

$(4.2.3)^{\circ}$: for each n, $||p_n^{\delta} \circ \mu^n - p_n \circ \mu^n||_{BV} \to 0$ as $\delta \to 0$.

$(4.2.4)^{\circ}$: $||p_n \circ \mu^n - f o \mu||_{BV} \to 0$ as $n \to \infty$

$(4.2.2)^{\circ}$, $(4.2.3)^{\circ}$ and $(4.2.4)^{\circ}$ imply

$(4.2.5)^{\circ}$ $||f^{\delta} o \mu - f o \mu|| \to 0$ as $\delta \to 0$

$(4.2.6)^{\circ}$: Since polynomials in m-variables are dense in $C^1(R(\mu))$ (See, 2^c, pp. 17), there exists a sequence $\{g_n\}$ of polynomials approximating f^{δ} and hence f, due to (E.2.5) i.e.

$$|| f o\mu - g_n o\mu || \to 0 \quad \text{as } n \to \infty.$$

$(4.2.7)^{\circ}$: "Any polynomials p on $R(\mu)$ is continuous" follows from the following observations :

(i) for every $\eta \in B(\nu)$, $||f o\eta||_{BV} = ||f o\nu||_{BV}$

(ii) for every $\eta \in B(\mu,\delta)$,

$$||f o\mu - f o\eta||_{BV} \leq ||f o\mu - g_n o\mu||_{BV} + ||g_n o\mu - g_n o\eta||_{BV}$$

$$+ ||g_n o\eta - f o\eta||_{BV} \to 0 \text{ as } n \to \infty \text{ and } \delta \to 0$$

Q.E.D.

4.3 <u>Notes:</u> Aumann and Shapley (1) have given a necessay and sufficient condition for scalar measure games to be in pNA, namely $f o\mu \in pNA \iff f$ is absolutely continuous on I, where $\mu \in NA^1$ and $f : I \to \mathbb{R}$ with $f(0) = 0$. A characterisation for the case when $\mu \in NA$ has been given by E. Kohlberg (12) in this connection also see (1). p. 74, proposition 9.1. Theorem 2.6 (Theorem B of A-S) provides a sufficient condition for vector measure games to be in pNA. But theorem 4.2 gives a complete characterisation of it.

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